

# AN ABSTRACT CHARACTERIZATION OF NONCOMMUTATIVE $\mathbb{P}^1$ -BUNDLES

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ABSTRACT. Let  $k$  be a field. We describe necessary and sufficient conditions for a  $k$ -linear abelian category to be a noncommutative  $\mathbb{P}^1$ -bundle over a pair of division rings over  $k$ . As an application, we prove that  $\mathbb{P}_n^1$ , Piontkovski's  $n$ th noncommutative projective line, is the noncommutative projectivization of an  $n$ -dimensional vector space.

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## 1. INTRODUCTION

Throughout this paper, we work over a field  $k$ .

**1.1. Motivation.** In noncommutative algebraic geometry, the primary objects of study are abelian categories which bear similarities to categories of coherent sheaves over schemes. When  $k$  is algebraically closed, a classification of categories like the category of coherent sheaves over a smooth projective curve has been achieved [2], [27]. In the case that  $k$  is not algebraically closed, there is a general theory of noncommutative curves due to Kussin [12], and those of genus zero have been studied extensively [11].

Recall that Kussin defines a noncommutative curve of genus zero to be a small  $k$ -linear noetherian, abelian, Ext-finite category  $\mathbf{C}$  with a Serre functor inducing the relevant form of Serre duality, an object of infinite length, and a tilting object. The curve  $\mathbf{C}$  is *homogenous* if for all simple objects  $\mathcal{S}$  in  $\mathbf{C}$ ,  $\text{Ext}_{\mathbf{C}}^1(\mathcal{S}, \mathcal{S}) \neq 0$ . Concrete examples of such curves are the noncommutative spaces  $\mathbb{P}^{nc}(M)$  defined by Van den Bergh [33], where  $D_0$  and  $D_1$  are division rings which are finite dimensional over  $k$ , and  $M$  is a  $k$ -central  $D_0 - D_1$ -bimodule of left-right dimension  $(1, 4)$  or  $(2, 2)$ . More specifically, if  $\mathbb{S}^{nc}(M)$  is the noncommutative symmetric algebra of  $M$  (see Section 3 for the definition of  $\mathbb{S}^{nc}(M)$ ), then  $\mathbb{P}^{nc}(M)$  is the category of finitely generated graded right  $\mathbb{S}^{nc}(M)$ -modules modulo those that are eventually zero.

In [18, Theorem 3.10] and [22, Theorem 3.1], the following converse is established:

**Theorem 1.1.** *Every homogeneous noncommutative curve of genus zero has the form  $\mathbb{P}^{nc}(M)$  for some  $k$ -central  $D_0 - D_1$ -bimodule of left-right dimension  $(1, 4)$  or  $(2, 2)$ .*

There exist many noncommutative spaces deserving of the appellation ‘noncommutative curve of genus zero’ that are not of the above form. In this paper, our goal is to characterize a generalization of the notion of noncommutative curve of genus zero which includes these other examples. Theorem 1.1 suggests how to generalize this notion: we should think of all spaces of the form  $\mathbb{P}^{nc}(M)$ , with  $M$  a bimodule over a pair of division rings not necessarily finite over  $k$ , and with finite left and right dimensions not necessarily equal to  $(1, 4)$  or  $(2, 2)$ , to be noncommutative curves of genus zero. Although one has to make some restrictions on  $M$  so that the category  $\mathbb{P}^{nc}(M)$  is homologically well behaved, once these restrictions are made, it turns out that one can describe necessary and sufficient conditions on a  $k$ -linear category to be equivalent to a category of this form. This constitutes our main result (see Section 4). In what follows, we refer to such spaces as noncommutative  $\mathbb{P}^1$ -bundles.

The notion of noncommutative  $\mathbb{P}^1$ -bundle we work with is broad enough to encompass examples not described using Kussin’s definition. In particular, both generic fibers of noncommutative ruled surfaces not finite over their centers ([23], [6]), and Piontkovski’s noncommutative projective lines ([24]), are noncommutative  $\mathbb{P}^1$ -bundles. The first class of examples is relevant to Artin’s conjecture [1], which loosely states that noncommutative surfaces which are infinite over their center are birationally ruled. In this context, our main theorem specializes to give conditions under which a category is equivalent to the generic fiber of a noncommutative ruled surface. The second class of examples implies that many noncommutative  $\mathbb{P}^1$ -bundles are non-noetherian, and leads us to work more generally within the

framework of coherent noncommutative algebraic geometry [25]. In this context, a noncommutative  $\mathbb{P}^1$ -bundle is a quotient category of the form

$$\mathbb{P}^{nc}(M) := \text{coh}\mathbb{S}^{nc}(M)/\text{tors}\mathbb{S}^{nc}(M)$$

where  $\text{coh}\mathbb{S}^{nc}(M)$  denotes the full subcategory of graded right  $\mathbb{S}^{nc}(M)$ -modules consisting of coherent modules and  $\text{tors}\mathbb{S}^{nc}(M)$  is the full subcategory of  $\text{coh}\mathbb{S}^{nc}(M)$  consisting of right-bounded modules.

**1.2. The main theorem in the finite-type case.** We now describe a special case of our main result, the so-called finite-type case, deferring a complete statement to Section 4 (Theorem 4.1). Its statement involves a collection of properties which may or may not hold for an arbitrary sequence

$$\underline{\mathcal{L}} := (\mathcal{L}_i)_{i \in \mathbb{Z}}$$

of objects in an abelian  $k$ -linear category  $\mathcal{C}$ . The properties we are interested in are as follows: for all  $i \in \mathbb{Z}$ ,

- $\text{End } \mathcal{L}_i =: D_i$  is a division ring finite-dimensional over  $k$ , and  $\dim_k D_i = \dim_k D_{i+2}$ ,
- $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i-1}) = 0 = \text{Ext}^1(\mathcal{L}_i, \mathcal{L}_{i-1})$ ,
- Let  $l_i = \dim_{D_{i+1}} \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$ . Then there is a short exact sequence

$$0 \rightarrow \mathcal{L}_i \rightarrow \mathcal{L}_{i+1}^{l_i} \rightarrow \mathcal{L}_{i+2} \rightarrow 0$$

such that the  $l_i$  morphisms defining the left arrow are left linearly independent.

- $\text{Ext}^1(\mathcal{L}_i, \mathcal{L}_j) = 0$  for all  $j \geq i$ .
- for all  $\mathcal{M}$  in  $\mathcal{C}$ ,  $\text{Hom}(\mathcal{L}_i, \mathcal{M})$  is a finite-dimensional  $D_i$ -module.
- $\underline{\mathcal{L}}$  is ample (see Section 2.3 for the definition of ampleness).

Finally, we need to introduce the following terminology regarding bimodules. We say a bimodule over a pair of division rings is of type  $(m, n)$  with  $m, n$  nonnegative integers if it has left-dimension  $m$  and right-dimension  $n$  or left-dimension  $n$  and right-dimension  $m$ .

Our main theorem specializes to the following:

**Theorem 1.2.** (*Finite-type case*) *The category  $\mathcal{C}$  has a sequence  $\underline{\mathcal{L}}$  satisfying the above six conditions if and only if*

$$\mathcal{C} \equiv \mathbb{P}^{nc}(M)$$

where  $M$  is a  $D_0 - D_1$ -bimodule of type  $(m, n) \neq (1, 1), (1, 2), (1, 3)$  such that  $\mathbb{S}^{nc}(M)$  is graded right-coherent.

The question of when the noncommutative symmetric algebra  $\mathbb{S}^{nc}(M)$  is graded right-coherent is a subject of current investigation by the author and others.

We call Theorem 1.2 the finite-type case as the rings  $\text{End } \mathcal{L}_i$  are assumed to be finite-dimensional over  $k$ . This hypothesis does not hold for generic fibers of noncommutative ruled surfaces, and is not necessary in general. The cost of removing the hypothesis is that we must then add other hypotheses to equate the left-dimension of a bimodule over division rings with the right-dimension of its right-dual.

As a consequence of Theorem 1.2 we show that the noncommutative projective lines,  $\mathbb{P}_n^1$ ,  $n \geq 2$ , studied in [24] and [29], are noncommutative  $\mathbb{P}^1$ -bundles. To describe this application in more detail we recall the definition of  $\mathbb{P}_n^1$ . To this

end, we need the following result of J.J. Zhang [35, Theorem 0.1]: every connected regular algebra of dimension two over  $k$  generated in degree one is isomorphic to an algebra of the form

$$(1-1) \quad A = k\langle x_1, \dots, x_n \rangle / (b)$$

where  $b = \sum_{i=1}^n x_i \sigma(x_{n-i})$  for some graded automorphism  $\sigma$  of the free algebra. Piontkovski shows that such rings are graded right-coherent [24, Theorem 4.3], and defines  $\mathbb{P}_n^1$  to be  $\text{cohproj} A$ , where  $A$  is as above, and  $\text{cohproj} A$  denotes the category of coherent graded right  $A$ -modules modulo those that are eventually zero. Piontkovski then proves [24, Theorem 1.5] that this category depends only on  $k$  and  $n$ , and that such categories share many homological properties with the category of coherent sheaves on (commutative)  $\mathbb{P}^1$ .

Our main result allows us to deduce the following

**Corollary 1.3.** *Let  $V$  be an  $n$ -dimensional vector space over  $k$ . Then there is an equivalence*

$$\mathbb{P}_n^1 \equiv \mathbb{P}^{nc}(V).$$

This corollary explains, in some sense,  $\mathbb{P}_n^1$ 's reliance only on  $n$  and  $k$  but not on the exact form of the relation  $b$  in (1-1).

As an immediate consequence of a corollary to Theorem 1.2 (Corollary 5.10) we also deduce the following related result:

**Corollary 1.4.** *The category  $\mathcal{C}$  has a sequence  $\underline{\mathcal{L}}$  such that the six conditions above are satisfied and  $D_i = k$  for all  $i$  if and only if there is a  $k$ -linear equivalence  $\mathcal{C} \equiv \mathbb{P}_n^1$  for some  $n$ .*

**1.3. Organization of the paper.** We now briefly describe the contents of this paper. In Section 2, after recalling the definition of  $\mathbb{Z}$ -algebras and noncommutative spaces of the form  $\text{Proj} A$  for  $A$  a  $\mathbb{Z}$ -algebra, we review basic results about graded coherence and ampleness from [25]. In Section 3, we recall the definition of noncommutative symmetric algebras and show, in Section 3.3, that they satisfy an Euler exact sequence, generalizing [6, Section 3.3] and [18, Lemma 3.7 and Proposition 3.8]. We then state our main result, Theorem 4.1, in Section 4. In Section 5, we study so-called linear sequences, in order to prove part of Theorem 4.1. The argument is a refinement of the argument used to prove Theorem 1.1.

Most of the rest of the paper consists of the proof of the remainder of Theorem 4.1, which entails proving that noncommutative  $\mathbb{P}^1$ -bundles are homologically well-behaved. In order to do this, we adapt the definition and study of relative local cohomology from [19] to our context (in Section 6 and Section 7), in order to show that noncommutative symmetric algebras are Gorenstein. There are two fundamental differences between the analysis in [19] and this paper: since our base is affine, the relevant functors we study are much easier to work with. On the other hand, since the noncommutative  $\mathbb{P}^1$ -bundles we study here are not noetherian, some of our proofs involve subtleties not encountered in the noetherian setting of [19].

Finally, in Section 8, we check that the spaces  $\mathbb{P}_n^1$  satisfy the hypotheses for our main result, allowing us to deduce Corollary 1.3.

## 2. PRELIMINARIES

**2.1.  $\mathbb{Z}$ -algebras and  $\text{Proj} A$ .** We recall the notion of a positively graded  $\mathbb{Z}$ -algebra and its graded modules, following [34, Section 2].

A  $\mathbb{Z}$ -algebra is a ring  $A$  with decomposition  $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$  into  $k$ -vector spaces, such that multiplication has the property  $A_{ij}A_{jk} \subset A_{ik}$  while  $A_{ij}A_{kl} = 0$  if  $j \neq k$ . Furthermore, for  $i \in \mathbb{Z}$ , there is a local unit  $e_i \in A_{ii}$ , such that if  $a \in A_{ij}$ , then  $e_i a = a = a e_j$ .  $A$  is *positively graded* if  $A_{ij} = 0$  for all  $i > j$ . In what follows, we will abuse terminology by saying ' $A$  is a  $\mathbb{Z}$ -algebra' if  $A$  is a positively graded  $\mathbb{Z}$ -algebra.

If  $A$  is a  $\mathbb{Z}$ -algebra then a graded right  $A$ -module  $M$  is a right  $A$ -module together with a decomposition  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  such that  $M_i A_{ij} \subset M_j$ ,  $e_i$  acts as a unit on  $M_i$  and  $M_i A_{jk} = 0$  if  $i \neq j$ .

We let  $\text{Gr}A$  denote the category of graded right  $A$ -modules (with obvious notion of homomorphism), and we note that it is a Grothendieck category (see [34, Section 2]).

For the remainder of Section 2.1, we assume that the  $\mathbb{Z}$ -algebra  $A$  satisfies the following conditions:

- for all  $i$ ,  $A_{ii}$  is a division ring over  $k$  for each  $i$ ,
- for all  $i$ ,  $A_{ii+1}$  is finite-dimensional over both  $A_{ii}$  and  $A_{i+1i+1}$ , and
- $A$  is generated in degree one, i.e. for all  $i$  and for  $j \geq i$ , the multiplication maps  $A_{ij} \otimes A_{jj+1} \rightarrow A_{ij+1}$  are surjective, and for  $j \geq i+1$ , the multiplication maps  $A_{ii+1} \otimes A_{i+1j} \rightarrow A_{ij}$  are surjective.

We let  $\text{Tors}A$  denote the full subcategory of  $\text{Gr}A$  consisting of modules whose elements are right bounded. Then the assumptions on  $A$  imply that  $\text{Tors}A$  is a Serre subcategory of  $\text{Gr}A$ , and there is a torsion functor  $\tau : \text{Gr}A \rightarrow \text{Tors}A$  which sends a module to its largest torsion submodule. Furthermore, since  $\text{Gr}A$  has enough injectives, it follows that if  $\pi : \text{Gr}A \rightarrow \text{Gr}A/\text{Tors}A =: \text{Proj}A$  is the quotient functor, then there exists a section functor  $\omega : \text{Proj}A \rightarrow \text{Gr}A$  which is right adjoint to  $\pi$ .

**2.2. Graded coherence.** We now review the basic facts about coherence from [25] which we will need in the sequel. For the rest of Section 2, we let  $A$  denote a  $\mathbb{Z}$ -algebra such that

- $A_{ii}$  is a division ring over  $k$  for each  $i$ , and
- $A_{ij}$  is finite-dimensional as a left  $A_{ii}$ -module and as a right  $A_{jj}$ -module.

We warn the reader that our grading convention and assumptions on  $A$  differ slightly from those appearing in [25].

We let  $P_i := \bigoplus_j A_{ij} = e_i A$ . We say that  $M \in \text{Gr}A$  is *finitely generated* if there is a surjection  $P \rightarrow M$  where  $P$  is a finite direct sum of modules of the form  $P_i$ . We say  $M$  is *coherent* if it is finitely generated and if for every homomorphism  $f : P \rightarrow M$  with  $P$  a direct sum of  $P_i$ 's,  $\ker f$  is finitely generated. We denote the full subcategory of  $\text{Gr}A$  consisting of coherent modules by  $\text{coh}A$ . By [25, Proposition 1.1],  $\text{coh}A$  is an abelian subcategory of  $\text{Gr}A$  closed under extensions.

We call  $A$  *graded right-coherent* (or just *coherent*) if the right modules  $P_j$  and  $S_j := P_j/P_{j>j}$  are coherent.

We let  $\text{tors}A$  denote the full subcategory of  $\text{coh}A$  consisting of right-bounded modules. One can check that this is a Serre subcategory of  $\text{coh}A$ . If  $A$  is graded coherent, we define

$$\text{cohproj}A := \text{coh}A/\text{tors}A,$$

which is abelian.

In the following result, which will be used in Section 7, we abuse notation by letting  $\pi : \text{coh}A \rightarrow \text{cohproj}A$  denote the quotient functor.

**Lemma 2.1.** *Suppose  $A$  is coherent and generated in degree one. Then*

- (1) *the inclusion functor  $\iota : \text{coh}A \rightarrow \text{Gr}A$  descends to an exact functor*

$$\underline{\iota} : \text{cohproj}A \rightarrow \text{Proj}A,$$

- (2) *the functor  $\underline{\iota}$  is faithful,*

- (3) *if  $N$  is an object in  $\text{coh}A$  such that  $\iota(N)$  is torsion-free, then the map*

$$\text{Hom}_{\text{cohproj}A}(\pi(M), \pi(N)) \rightarrow \text{Hom}_{\text{Proj}A}(\underline{\iota}\pi(M), \underline{\iota}\pi(N))$$

*induced by  $\underline{\iota}$  is surjective, and*

- (4) *if  $M$  and  $N$  are objects in  $\text{coh}A$  such that  $\iota(M)$  and  $\iota(N)$  are torsion-free, then the map*

$$(2-1) \quad \text{Ext}_{\text{cohproj}A}^1(\pi(M), \pi(N)) \rightarrow \text{Ext}_{\text{Proj}A}^1(\underline{\iota}\pi(M), \underline{\iota}\pi(N))$$

*induced by  $\underline{\iota}$  is an injection of  $\text{End}(\pi(N)) - \text{End}(\pi(M))$ -bimodules.*

*Proof.* By hypothesis,  $A$  satisfies the properties listed in Section 2.1, so that  $\text{Proj}A$  is well-defined.

By [25, Proposition 1.1], the inclusion  $\iota : \text{coh}A \rightarrow \text{Gr}A$  makes  $\text{coh}A$  an abelian subcategory of  $\text{Gr}A$ . Furthermore, if  $M \in \text{tors}A$ , then  $\iota(M) \in \text{Tors}A$ . Therefore, by [8, Corollaire 2, p. 368],  $\iota$  induces a functor  $\underline{\iota} : \text{cohproj}A \rightarrow \text{Proj}A$  which is exact by [8, Corollaire 3, p. 369]. Part (1) of the lemma follows.

We next show that  $\underline{\iota}$  is faithful. To this end, we recall that, by definition of  $\underline{\iota}$ ,  $\underline{\iota}(\pi(M)) = \pi(\iota(M))$ , and if  $f \in \text{Hom}_{\text{cohproj}A}(\pi(M), \pi(N))$ , then we may identify  $f$  with a map  $f' \in \text{Hom}_{\text{coh}A}(M', N/N')$  where  $M' \subset M$  is coherent,  $M/M' \in \text{tors}A$ , and  $N' \subset N$  is in  $\text{tors}A$ . Furthermore, by the proof of [8, Corollaire 2, p. 368],  $\underline{\iota}(f) \in \text{Hom}_{\text{Proj}A}(\pi(\iota(M)), \pi(\iota(N)))$  corresponds to  $\iota(f') \in \text{Hom}_{\text{Gr}A}(\iota(M'), \iota(N)/\iota(N'))$ . Therefore, if  $\underline{\iota}(f) = 0$ , it follows that  $\pi(\iota(f')) = 0$  which implies that the image of  $\iota(f')$  is torsion. Since  $M'$  is coherent, the image of  $\iota(f')$  is right-bounded, so that the image of  $f'$  is in  $\text{tors}A$ . It follows that  $\pi(f') = 0$ , so that  $f = 0$ .

Now we prove (3). Suppose  $N \in \text{coh}A$  is such that  $\iota(N)$  is torsion-free. We claim that if  $M' \subset \iota(M)$  is such that  $\iota(M)/M'$  is in  $\text{Tors}A$ , then  $M'$  is coherent. It will follow that  $\iota$  induces a bijection  $\text{Hom}_{\text{coh}A}(M', N) \rightarrow \text{Hom}_{\text{Gr}A}(\iota(M'), \iota(N))$ , from which (3) will follow. We now prove the claim. Since  $M$  is coherent, it is finitely generated, so that  $\iota(M)$  and  $M'$  are equal in high degree. Thus, by our assumptions on  $A$ ,  $\iota(M)/M'$  is coherent. It follows that  $M'$  is coherent as desired.

Finally, we prove (4). We first recall that the group  $\text{Ext}_{\text{cohproj}A}^1(\pi(M), \pi(N))$  is defined as Yoneda's extension group, while the group  $\text{Ext}_{\text{Proj}A}^1(\underline{\iota}\pi(M), \underline{\iota}\pi(N))$  defined in terms of injective resolutions is isomorphic to Yoneda's extension group as a bimodule [17, Section VII.7].

By (1), the functor  $\underline{\iota}$  applied to an exact sequence in  $\text{cohproj}A$  is exact in  $\text{Proj}A$ , and hence determines an element of  $\text{Ext}_{\text{Proj}A}^1(\underline{\iota}\pi(M), \underline{\iota}\pi(N))$ . The fact that this induces a morphism between extension groups follows immediately from functoriality of  $\underline{\iota}$ .

Since  $\underline{\iota}$  is exact and additive it preserves pullbacks and pushouts. It follows that  $\underline{\iota}$  induces an additive function between extension groups, and that the assignment is compatible with bimodule structures, as one can check.

To prove injectivity of (2-1), suppose  $\underline{\iota}$  applied to an extension

$$0 \rightarrow \pi(N) \rightarrow \pi(P) \rightarrow \pi(M) \rightarrow 0$$

in  $\text{cohproj} A$  maps to a trivial extension in  $\text{Proj} A$ . We then have an isomorphism  $\underline{\iota}\pi(P) \rightarrow \underline{\iota}\pi(M \oplus N)$  in  $\text{Proj} A$ . Since  $\iota(M \oplus N)$  is torsion-free by hypothesis, part (3) implies that the isomorphism must be induced by an isomorphism  $\pi(P) \rightarrow \pi(M \oplus N)$ , and the result follows.  $\square$

**2.3. Ampleness.** We let  $\mathbf{C}$  denote a  $k$ -linear category, we let  $\underline{\mathcal{E}} = (\mathcal{E}_i)_{i \in \mathbb{Z}}$  denote a sequence of objects in  $\mathbf{C}$  such that  $\text{Hom}(\mathcal{E}_i, \mathcal{E}_i)$  is a division ring, and (in this section) we assume that for all  $\mathcal{M} \in \mathbf{C}$ , the dimension of  $\text{Hom}_{\mathbf{C}}(\mathcal{E}_i, \mathcal{M})$  is finite as a right  $\text{End}(\mathcal{E}_i)$ -module.

We call  $\underline{\mathcal{E}}$

- *projective* if for every surjection  $f : \mathcal{M} \rightarrow \mathcal{N}$  in  $\mathbf{C}$  there exists an integer  $n$  such that  $\text{Hom}_{\mathbf{C}}(\mathcal{E}_{-i}, f)$  is surjective for all  $i > n$ , and
- *ample* if it is projective, and if for every  $\mathcal{M} \in \mathbf{C}$  and every  $m \in \mathbb{Z}$  there exists a surjection

$$\bigoplus_{j=1}^s \mathcal{E}_{-i_j} \rightarrow \mathcal{M}$$

for some  $i_1, \dots, i_s$  with  $i_j \geq m$  for all  $j$ .

We will need the following special case of [25, Proposition 2.3(ii) and Theorem 2.4]:

**Theorem 2.2.** *If  $\underline{\mathcal{E}}$  is an ample sequence and  $A(\underline{\mathcal{E}})$  denotes the  $\mathbb{Z}$ -algebra with  $A_{ij} := \text{Hom}_{\mathbf{C}}(\mathcal{E}_{-j}, \mathcal{E}_{-i})$  for  $i \leq j$  and  $A_{ij} = 0$  otherwise, then  $A(\underline{\mathcal{E}})$  is coherent, and there is an equivalence*

$$\mathbf{C} \equiv \text{cohproj} A(\underline{\mathcal{E}}).$$

### 3. NONCOMMUTATIVE SYMMETRIC ALGEBRAS

Let  $R$  and  $S$  be noetherian  $k$ -algebras. In this section, following [33], we define the noncommutative symmetric algebra of certain  $R - S$ -bimodules. Some of the exposition is adapted from [22].

**3.1. Bimodules.** We assume throughout this section that  $N$  is an  $R - S$ -bimodule which is finitely generated projective as both a left  $R$ -module and as a right  $S$ -module. We recall that the *right dual* of  $N$ , denoted  $N^*$ , is the  $S - R$ -bimodule whose underlying set is  $\text{Hom}_S(N_S, S)$ , with action

$$(a \cdot \psi \cdot b)(n) = a\psi(bn)$$

for all  $\psi \in \text{Hom}_S(N_S, S)$ ,  $a \in S$  and  $b \in R$ .

The *left dual* of  $N$ , denoted  ${}^*N$ , is the  $S - R$ -bimodule whose underlying set is  $\text{Hom}_R({}_R N, R)$ , with action

$$(a \cdot \phi \cdot b)(n) = \phi(na)b$$

for all  $\phi \in \text{Hom}_R({}_R N, R)$ ,  $a \in S$  and  $b \in R$ . This assignment extends to morphisms between  $R - S$ -bimodules in the obvious way.

We set

$$N^{i*} := \begin{cases} N & \text{if } i = 0, \\ (N^{i-1*})^* & \text{if } i > 0, \\ {}^*(N^{i+1*}) & \text{if } i < 0. \end{cases}$$

In general,  $N$  may not be isomorphic to  $N^{**}$  or  ${}^{**}N$  [28, Section 6.4]. Furthermore, although  $N^*$  (resp.  ${}^*N$ ) is finitely generated projective on the left (resp. finitely generated projective on the right), it is not clear that  $N^*$  is finitely generated

projective on the right (resp. finitely generated projective on the left). Therefore, we make the following

*Definition 3.1.* We say  $N$  is *admissible* if  $N^{i*}$  is finitely generated projective on each side for all  $i \in \mathbb{Z}$ . We say  $N$  is *2-periodic* if  $N$  is admissible,  $N^{i*}$  is free on each side, and the left rank of  $N^{i*}$  equals the right rank of  $N^{i+1*}$ .

We remark that if  $R$  and  $S$  are finite dimensional simple rings over  $k$ , then  $N$  is automatically 2-periodic. If  $R$  and  $S$  are fields and  $N$  is of type  $(2, 2)$ , then  $N$  is 2-periodic [6, Lemma 3.4]. Finally, if  $R = S$  is a perfect field and  $N$  has finite left and right dimension, then  $N$  is 2-periodic [9, Proposition 4.3].

If  $R$  and  $S$  are division rings and  $N$  has finite left-dimension  $m$  and finite right-dimension  $n$ , we say  $N$  has *left-right dimension*  $(m, n)$ . In this situation, we let  $\text{ldim } N$  denote the dimension of  $N$  over  $R$  and we let  $\text{rdim } N$  denote the dimension of  $N$  over  $S$ .

Let

$$R_i = \begin{cases} R & \text{if } i \text{ is even, and} \\ S & \text{if } i \text{ is odd.} \end{cases}$$

In what follows, all unadorned tensor products will be over  $R_i$ .

We recall that, if  $N$  is admissible, then for each  $i$ , both pairs of functors

$$(3-1) \quad (- \otimes_{R_i} N^{i*}, - \otimes_{R_{i+1}} N^{i+1*})$$

and

$$(3-2) \quad (- \otimes_{R_i} {}^*(N^{i+1*}), - \otimes_{R_{i+1}} N^{i+1*})$$

between the category of right  $R_i$ -modules and the category of right  $R_{i+1}$ -modules have canonical adjoint structures.

We denote the images of the units applied to  $R_i$  by  $Q_i$  and  $Q'_i$ , respectively. If  $R$  and  $S$  are division rings,  $N$  is 2-periodic,  $\{\phi_1, \dots, \phi_n\}$  is a right-basis for  $N^{i*}$  and  $\{\phi_1^*, \dots, \phi_n^*\}$  is a corresponding left dual basis for  $N^{i+1*}$ , then the canonical unit map from  $R_i$  to  $Q_i$  maps 1 to  $\sum_i \phi_i \otimes \phi_i^*$  (see [18, Section 2]). In particular, the later element is  $R_i$ -central. We will employ this fact without comment in the sequel.

**3.2. The definition of  $\mathbb{S}^{nc}(M)$ .** We now recall (from [33]) the definition of the noncommutative symmetric algebra of an admissible  $R$ - $S$ -bimodule  $N$ . The *noncommutative symmetric algebra of  $N$* , denoted  $\mathbb{S}^{nc}(N)$ , is the  $\mathbb{Z}$ -algebra  $\bigoplus_{i,j \in \mathbb{Z}} A_{ij}$

with components defined as follows:

- $A_{ii} = R$  for  $i$  even,
- $A_{ii} = S$  for  $i$  odd, and
- $A_{ii+1} = N^{i*}$ .

In order to define  $A_{ij}$  for  $j > i + 1$ , we introduce some notation: we define  $T_{ii+1} := A_{ii+1}$ , and, for  $j > i + 1$ , we define

$$T_{ij} := A_{ii+1} \otimes A_{i+1i+2} \otimes \cdots \otimes A_{j-1j}.$$

We let  $R_{ii+1} := 0$ ,  $R_{ii+2} := Q_i$ ,

$$R_{ii+3} := Q_i \otimes N^{i+2*} + N^{i*} \otimes Q_{i+1},$$

and, for  $j > i + 3$ , we let

$$R_{ij} := Q_i \otimes T_{i+2j} + T_{ii+1} \otimes Q_{i+1} \otimes T_{i+3j} + \cdots + T_{ij-2} \otimes Q_{j-2}.$$



- For  $j > i + 1$ , we define  $A_{ij}$  as the quotient  $T_{ij}/R_{ij}$ .

Multiplication in  $\mathbb{S}^{nc}(N)$  is defined as follows:

- if  $x \in A_{ij}$  and  $y \in A_{jk}$ , with either  $i = j$  or  $j = k$ , then  $xy$  is induced by the usual scalar action,
- otherwise, if  $i < j < k$ , we have

$$\begin{aligned} A_{ij} \otimes A_{jk} &= \frac{T_{ij}}{R_{ij}} \otimes \frac{T_{jk}}{R_{jk}} \\ &\cong \frac{T_{ik}}{R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk}}. \end{aligned}$$

Since  $R_{ij} \otimes T_{jk} + T_{ij} \otimes R_{jk}$  is a submodule of  $R_{ik}$ , we may define multiplication  $A_{ij} \otimes A_{jk} \rightarrow A_{ik}$  as the canonical epimorphism.

**3.3. Euler sequences.** In this section, we assume  $D_0$  and  $D_1$  are division rings over  $k$ , we let  $M$  be a  $k$ -central  $D_0 - D_1$ -bimodule such that  $M$  is 2-periodic and not of type  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$ , and we let  $A = \mathbb{S}^{nc}(M)$ . Our main goal in this section, Corollary 3.5, is to prove that the trivial module in  $\text{Gr}A$  has the expected resolution. We then derive some consequences, which we will need in the sequel.

**Lemma 3.2.** *Let  $j \in \mathbb{Z}$  and suppose  $M^{j*}$  has left-right dimension  $(n, m)$  with  $m \geq 2$ . If  $v \in M^{j*}$  has the property that  $v \otimes g \in Q_j$  for some nonzero  $g \in M^{j+1*}$ , then  $v = 0$ .*

*Proof.* Let  $\{\phi_1, \dots, \phi_m\}$  denote a right basis for  $M^{j*}$  so that  $\{\phi_1^*, \dots, \phi_m^*\}$  (the right-dual basis) is a left basis for  $M^{j+1*}$ . Let  $v = \sum_l \phi_l a_l$  and suppose  $v \neq 0$ . We have

$$\begin{aligned} (\sum_l \phi_l \otimes \phi_l^*)c &= \sum_l \phi_l a_l \otimes g \\ &= \sum_l \phi_l \otimes a_l g. \end{aligned}$$

Therefore, for all  $l$ ,  $a_l g = \phi_l^* c$ . Since there exists an  $l$  such that  $a_l \neq 0$ , we have  $c \neq 0$ . It follows that all  $a_l$  are nonzero. Since  $m \geq 2$ , we thus have  $a_1^{-1} \phi_1^* = a_2^{-1} \phi_2^*$ , which is a contradiction.  $\square$

The following generalizes [6, Lemma 3.5]:

**Proposition 3.3.** *Let  $j \in \mathbb{Z}$  and suppose  $M^{j*}$  has left-right dimension  $(n, m)$  with  $m, n \geq 2$ . If  $v \in T_{ij+1}$  has the property that  $v \otimes g \in R_{ij+2}$  for some nonzero  $g \in M^{j+1*}$ , then  $v \in R_{ij+1}$ .*

*Proof.* Throughout the proof, we let  $\{f_1, \dots, f_n\}$  be a left basis of  $M^{j*}$  so that  $\{^*f_1, \dots, ^*f_n\}$  (the left-dual basis) is a right basis of  $M^{j-1*} \cong ^*(M^{j*})$ . Similarly, we let  $\{\phi_1, \dots, \phi_m\}$  denote a right basis for  $M^{j*}$  so that  $\{\phi_1^*, \dots, \phi_m^*\}$  (the right-dual basis) is a left basis for  $M^{j+1*}$ .

We proceed by induction on  $j - i$ . First, if  $i = j$ , then the result follows from Lemma 3.2.

Next, we assume  $j = i + 1$ . If  $g = \sum_r b_r \phi_r^*$ , then there exists an  $h \in M^{j-1*}$  such that

$$v \otimes \sum_r b_r \phi_r^* - h \otimes \sum_s \phi_s \otimes \phi_s^* \in R_{ij+1} \otimes M^{j+1*}.$$

It follows that

$$vb_r - h \otimes \phi_r \in R_{ij+1}$$

for all  $r$ . Thus

$$vb_r - h \otimes \phi_r = c_r \left( \sum_l {}^*f_l \otimes f_l \right)$$

for all  $r$ . Without loss of generality, assume  $b_1 \neq 0$ . Then we deduce

$$h \otimes \phi_1 b_1^{-1} b_2 + c_1 b_1^{-1} b_2 \left( \sum_l {}^*f_l \otimes f_l \right) = h \otimes \phi_2 + c_2 \left( \sum_l {}^*f_l \otimes f_l \right).$$

Thus,

$$h \otimes (\phi_1 b_1^{-1} b_2 - \phi_2) + (c_1 b_1^{-1} b_2 - c_2) \left( \sum_l {}^*f_l \otimes f_l \right) = 0.$$

Since the  $\phi_i$  are right-independent,  $\psi := \phi_1 b_1^{-1} b_2 - \phi_2 \neq 0$ . Therefore,  $h \otimes \psi \in R_{ii+2}$ . If we write  $\psi = \sum d_q f_q$ , then we have

$$h \otimes \psi = h \otimes \sum_q d_q f_q = e \sum {}^*f_l \otimes f_l$$

and, therefore,  $hd_q = e^* f_q$  for all  $q$ . If  $d_q = 0$  for some  $q$ , then  $e = 0$  so that  $hd_q = 0$  for all  $q$ , which is a contradiction unless  $h = 0$ . Otherwise,  $d_1 \neq 0$  and  $d_2 \neq 0$ , which implies that since  $hd_1 = e^* f_1$  and  $hd_2 = e^* f_2$ , we have  $e^* f_1 d_1^{-1} = e^* f_2 d_2^{-1}$ . If  $e \neq 0$ , this contradicts the right-independence of the  ${}^*f_i$ 's. Therefore,  $e = 0$  so that  $h = 0$  which implies that  $v \in R_{ij+1}$  as desired.

Now we assume  $j > i + 1$  and let  $g = \sum_r b_r \phi_r^*$  as above. Then

$$v \otimes g - h \otimes \sum \phi_s \otimes \phi_s^* \in R_{ij+1} \otimes M^{j+1*}$$

for some  $h \in T_{ij}$  so that  $vb_r - h \otimes \phi_r \in R_{ij+1}$  for all  $r$ . Thus, for all  $r$ , we have

$$vb_r - h \otimes \phi_r - h_r \otimes \left( \sum {}^*f_l \otimes f_l \right) \in R_{ij} \otimes M^{j*}$$

for some  $h_r \in T_{ij-1}$ . Suppose, without loss of generality, that  $b_1 \neq 0$ . Then

$$(3-3) \quad vb_1 - h \otimes \phi_1 - h_1 \otimes \left( \sum {}^*f_l \otimes f_l \right) \in R_{ij} \otimes M^{j*}$$

while

$$(3-4) \quad vb_2 - h \otimes \phi_2 - h_2 \otimes ({}^*f_l \otimes f_l) \in R_{ij} \otimes M^{j*}.$$

Therefore, if we multiply (3-3) by  $b_1^{-1} b_2$  on the right and subtract (3-4), we deduce that

$$h \otimes (\phi_2 - \phi_1 b_1^{-1} b_2) + (h_2 - h_1 b_1^{-1} b_2) \otimes \left( \sum {}^*f_l \otimes f_l \right) \in R_{ij} \otimes M^{j*}.$$

Since  $\phi_2 - \phi_1 b_1^{-1} b_2 \neq 0$ , induction implies that  $h \in R_{ij}$ , and it follows that  $v \in R_{ij+1}$ .  $\square$

**Theorem 3.4.** *For all  $i \leq j$ , the canonical complex*

$$(3-5) \quad 0 \rightarrow A_{ij} \otimes Q_j \rightarrow A_{ij+1} \otimes M^{j+1*} \rightarrow A_{ij+2} \rightarrow 0$$

*is exact.*

*Proof.* There are two fundamentally different cases to consider. First, if  $M$  has left-right dimension  $(1, n)$  for  $n \geq 4$ , then the proofs of [18, Lemma 3.7] and [18, Proposition 3.8] still work in our context. Thus, we may suppose that  $M$  is of type  $(n, m)$  with both  $n$  and  $m \geq 2$ . In this case, suppose without loss of generality, that  $M^{j*}$  has left-right dimension  $(n, m)$ . We show that (3-5) is exact on the left. In order to prove this, it suffices to check that

$$(3-6) \quad R_{ij+1} \otimes M^{j+1*} \cap T_{ij} \otimes Q_j = R_{ij} \otimes Q_j.$$

As in the proof of Proposition 3.3, we let  $\{f_1, \dots, f_n\}$  be a left basis of  $M^{j*}$  so that  $\{^*f_1, \dots, ^*f_n\}$  (the left-dual basis) is a right basis of  $M^{j-1*} \cong ^*(M^{j*})$ . Similarly, we let  $\{\phi_1, \dots, \phi_m\}$  denote a right basis for  $M^{j*}$  so that  $\{\phi_1^*, \dots, \phi_m^*\}$  (the right-dual basis) is a left basis for  $M^{j+1*}$ .

First, we assume  $j = i$ . Then both sides of (3-6) are zero, as desired.

Next, we assume  $j = i + 1$ . Suppose  $v \otimes \sum_p \phi_p \otimes \phi_p^*$  is an element of the left-hand side of (3-6). Then we have an equality

$$v \otimes \sum_p \phi_p \otimes \phi_p^* = a \left( \sum_q ^*f_q \otimes f_q \right) \otimes \sum_r a_r \phi_r^*$$

for some scalars  $a, a_1, \dots, a_m$ , so that  $v \otimes \phi_l = aa_l \sum_q ^*f_q \otimes f_q$  for all  $l$ . Without loss of generality, we may assume  $\phi_1 = f_1$  so that  $aa_1 = 0$ . If  $a = 0$  then  $v \otimes \phi_l = 0$  for all  $l$  which implies that  $v = 0$ . Otherwise,  $a_1 = 0$ , so that  $v \otimes f_1 = 0$ . By Proposition 3.3,  $v = 0$  and the assertion follows in this case.

Now suppose  $j > i + 1$ . Once again, suppose  $v \otimes \sum_p \phi_p \otimes \phi_p^*$  is in the left-hand side of (3-6). Then there exists a  $w \in R_{ij} \otimes M^{j*}$ , a  $u \in T_{ij-2}$  and scalars  $b_1, \dots, b_m$  such that

$$v \otimes \sum_p \phi_p \otimes \phi_p^* = \sum_q w \otimes a_q \phi_q^* + u \otimes \left( \sum_r ^*f_r \otimes f_r \right) \otimes \sum_s b_s \phi_s^*.$$

It follows that, for all  $1 \leq l \leq m$ , we have

$$(3-7) \quad v \otimes \phi_l - ub_l \otimes \sum_r ^*f_r \otimes f_r \in R_{ij} \otimes M^{j*}.$$

If  $b_l = 0$  for some  $l$ , Proposition 3.3 thus implies  $v \in R_{ij}$  as desired. Thus, assume that  $b_l \neq 0$  for all  $l$ . Since, without loss of generality, we may take  $\phi_1 = f_1$ , (3-7) implies that  $v - ub_1 \otimes ^*f_1 \in R_{ij}$  and  $ub_1 \otimes ^*f_2 \in R_{ij}$ . The second inclusion and Proposition 3.3 implies that  $u \in R_{ij-1}$  so that the first inclusion implies that  $v \in R_{ij}$  as desired.  $\square$

**Corollary 3.5.** *For all  $k \in \mathbb{Z}$ , multiplication in  $A$  induces an exact sequence of right  $A$ -modules*

$$0 \rightarrow Q_{i-2} \otimes e_i A \rightarrow A_{i-2i-1} \otimes e_{i-1} A \rightarrow e_{i-2} A \rightarrow e_{i-2} A / e_{i-2} A_{\geq i-1} \rightarrow 0.$$

*Proof.* The only nontrivial part of the proof is to show the sequence is exact on the left. This can be checked by counting right-dimensions, which can be deduced from the exactness of (3-5). More precisely, one can prove, using (3-5) and induction on the difference of indices, that the right-dimension of  $A_{i-2j}$  equals the right-dimension of  $A_{ij+2}$ , and the right-dimension of  $A_{i-2i-1} \otimes A_{i-1j}$  equals the right-dimension of  $A_{ij+1} \otimes A_{j+1j+2}$ . Using these facts, the result follows immediately from (3-5). We leave the details to the reader.  $\square$

We will need the following result for our proof of Theorem 7.1.

**Lemma 3.6.** *If  $x \in A_{ij+1}$  is such that  $xy = 0$  for all  $y \in A_{j+1j+2}$ , then  $x = 0$ .*

*Proof.* If  $j+1 \leq i$  the result is trivial, so suppose  $j+1 > i$ . If  $A_{j+1j+2}$  has left-right dimension  $(n, m)$  with  $m, n \geq 2$ , the result follows from Proposition 3.3. Similarly, if  $A_{j+1j+2}$  has left-right dimension  $(1, n)$  with  $n \geq 4$ , the result follows from [18, Lemma 3.7].

It remains to prove the result in case  $A_{j+1j+2}$  has left-right dimension  $(n, 1)$  with  $n \geq 4$ . We let  $\{\phi_1, \dots, \phi_m\}$  denote a right basis for  $A_{jj+1}$  so that  $\{\phi_1^*, \dots, \phi_m^*\}$  (the right-dual basis) is a left basis for  $M^{j+1*}$ . Since  $xy = 0$  for all  $y \in A_{j+1j+2}$ , Theorem 3.4 implies that in  $A_{ij+1} \otimes M^{j+1*}$ ,  $x \otimes \phi_1^* = \sum_l z \phi_l \otimes \phi_l^*$  for some  $z \in A_{ij}$ . It follows that  $x = z\phi_1$ ,  $0 = z\phi_2$  and  $0 = z\phi_3$ . By [18, Lemma 3.7],  $z = 0$ , so that  $x = 0$ .  $\square$

Since in this section  $A$  satisfies the hypotheses in Section 2.1, we may form the category  $\text{Proj}A$ . As in Section 2.1, we let  $\pi : \text{Gr}A \rightarrow \text{Proj}A$  denote the quotient functor, and we let  $\mathcal{A}_i := \pi(e_i A)$ . Applying  $\pi$  to the Euler sequence from Corollary 3.5 yields, for each  $i \in \mathbb{Z}$ , an exact sequence

$$(3-8) \quad 0 \rightarrow \mathcal{A}_i \rightarrow \mathcal{A}_{i-1}^{d_i} \rightarrow \mathcal{A}_{i-2} \rightarrow 0$$

in  $\text{Proj}A$ , where  $d_i$  is the right-dimension of  $A_{i-2i-1}$ .

**Proposition 3.7.** *Suppose  $a \in \text{End}(\mathcal{A}_i)$  is induced by left-multiplication by  $\alpha \in A_{ii}$ . Then there exist  $f_a \in \text{End}(\mathcal{A}_{i-1}^{d_i})$  and  $a' \in \text{End}(\mathcal{A}_{i-2})$  such that the diagram*

$$(3-9) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{A}_i & \rightarrow & \mathcal{A}_{i-1}^{d_i} & \rightarrow & \mathcal{A}_{i-2} \rightarrow 0 \\ & & a \downarrow & & \downarrow f_a & & \downarrow a' \\ 0 & \rightarrow & \mathcal{A}_i & \rightarrow & \mathcal{A}_{i-1}^{d_i} & \rightarrow & \mathcal{A}_{i-2} \rightarrow 0, \end{array}$$

whose horizontals are (3-8), commutes. Moreover,  $a'$  is induced by left-multiplication by  $\alpha \in A_{i-2i-2} = A_{ii}$ .

*Proof.* We prove the existence of the desired morphisms in the graded module category, using Corollary 3.5. To this end, consider the partial square

$$\begin{array}{ccc} Q_{i-2} \otimes e_i A & \rightarrow & A_{i-2i-1} \otimes e_{i-1} A \\ \downarrow & & \\ Q_{i-2} \otimes e_i A & \rightarrow & A_{i-2i-1} \otimes e_{i-1} A \end{array}$$

whose horizontals are from Corollary 3.5 and whose vertical is induced by left-multiplication by  $\alpha$ . Suppose  $\{\phi_j\}_{j=1}^n$  is a right-basis for  $A_{i-2,i-1}$ , and let  $\{\phi_j^*\}_{j=1}^n$  be the dual left-basis for  $A_{i-1,i}$ . An element of the form  $(\sum_j \phi_j \otimes \phi_j^*) \otimes z$  from the upper-left goes to  $\sum_j \phi_j \otimes \phi_j^* z$  in the upper-right and  $\sum_j \phi_j \otimes \phi_j^* \alpha z$  on the lower-right. On the other hand, since  $\{\phi_j^*\}_{j=1}^n$  is a left basis for  $A_{i-1,i}$ , we may write  $\phi_j^* \alpha = \sum_l \alpha_{jl} \phi_l^*$  for appropriate scalars  $\alpha_{jl} \in A_{i-1,i-1}$ . Thus, if we define  $f_a$  as the map induced by sending  $\sum_j \phi_j \otimes z_j$  to  $\sum_j \phi_j \otimes (\sum_l \alpha_{jl} z_l)$ , then the left square of (3-9) commutes.

Next, we claim that if  $a'$  is induced by left-multiplication by  $\alpha$ , then the right-square of (3-9) commutes. As above, it suffices to show the square

$$\begin{array}{ccc} A_{i-2i-1} \otimes e_{i-1}A & \longrightarrow & e_{i-2}A \\ \downarrow & & \downarrow \\ A_{i-2i-1} \otimes e_{i-1}A & \longrightarrow & e_{i-2}A \end{array}$$

whose left-vertical is the map defined in the previous paragraph, whose right-vertical is induced by left-multiplication by  $\alpha$ , and whose horizontals are induced by multiplication in  $A$ , commutes. An element  $\sum_j \phi_j \otimes z_j$  in the top-left goes to  $\sum_j \phi_j z_j$  in the top-right. On the other hand, it goes to  $\sum_{j,l} \phi_j \alpha_{jl} z_l$  in the bottom-right. Thus, it suffices to show that  $\sum_j \alpha \phi_j z_j = \sum_{j,l} \phi_j \alpha_{jl} z_l$ . This follows from [9, Section 3.2].  $\square$

#### 4. STATEMENT OF THE MAIN THEOREM

In this section,  $\mathcal{C}$  will denote a  $k$ -linear abelian category. Our main theorem will involve a collection of properties describing a sequence

$$\underline{\mathcal{L}} := (\mathcal{L}_i)_{i \in \mathbb{Z}}$$

of objects in  $\mathcal{C}$ . The properties we are interested in are as follows: for all  $i \in \mathbb{Z}$ ,

- (1) End  $\mathcal{L}_i =: D_i$  is a division ring with  $k$  in its center,
- (2)  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i-1}) = 0 = \text{Ext}^1(\mathcal{L}_i, \mathcal{L}_{i-1})$ ,
- (3)  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$  is finite-dimensional as both a left  $D_{i+1}$ -module and a right  $D_i$ -module. The left and right dimensions are denoted  $l_i$  and  $r_i$ .
- (4) There is a short exact sequence, which we call the  *$i$ th Euler sequence*,

$$0 \rightarrow \mathcal{L}_i \rightarrow \mathcal{L}_{i+1}^{l_i} \rightarrow \mathcal{L}_{i+2} \rightarrow 0$$

such that the  $l_i$  morphisms defining the left arrow are left linearly independent.

- (5) The canonical one-to-one  $k$ -algebra map  $\Phi_i : D_i \rightarrow D_{i+2}$  (defined in Lemma 4.4 using (4)) is an isomorphism.
- (6)  $\text{Ext}^1(\mathcal{L}_i, \mathcal{L}_j) = 0$  for all  $j \geq i$ .
- (7) There is an equality  $l_i = r_{i-1}$ .
- (8) For all  $\mathcal{M}$  in  $\mathcal{C}$ ,  $\text{Hom}(\mathcal{L}_i, \mathcal{M})$  is a finite-dimensional right  $D_i$ -module.
- (9) The sequence  $\underline{\mathcal{L}}$  is ample.

From the sequence  $\underline{\mathcal{L}}$  we can form the  $\mathbb{Z}$ -algebra  $H$  with  $H_{ij} = \text{Hom}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$  and with multiplication induced by composition.

We prove the following result:

**Theorem 4.1.** *Let  $\underline{\mathcal{L}}$  denote a sequence of objects in a  $k$ -linear abelian category  $\mathcal{C}$ .*

- *If  $\underline{\mathcal{L}}$  satisfies (1)-(5), then  $H_{ii+1}$  is admissible and there is a  $k$ -linear  $\mathbb{Z}$ -algebra homomorphism*

$$(4-1) \quad \mathbb{S}^{nc}(H_{01}) \rightarrow H$$

*which is an isomorphism in degrees zero and one.*

- *If  $\underline{\mathcal{L}}$  satisfies (1)-(6), then the homomorphism (4-1) is an epimorphism.*
- *If  $\underline{\mathcal{L}}$  satisfies (1)-(7), then the homomorphism (4-1) is an isomorphism, and  $H_{ii+1}$  is 2-periodic and not of type (1, 1), (1, 2) or (1, 3).*

- The category  $\mathbf{C}$  has a sequence  $\underline{\mathcal{L}}$  satisfying (1)-(9) if and only if there is a  $k$ -linear equivalence  $\mathbf{C} \equiv \mathbb{P}^{nc}(M)$  with  $M$  a 2-periodic bimodule not of type  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$  such that  $\mathbb{S}^{nc}(M)$  is graded coherent.

By Theorem 1.1 and Theorem 4.1, Kussin's noncommutative curves of genus zero [11] have sequences satisfying (1)-(9). In addition, although the noncommutative curves studied in [6] are noetherian, they are not noncommutative curves of genus zero in the sense of [11] since they are not necessarily Ext-finite. Nevertheless, Theorem 4.1 implies they also have a sequence  $\underline{\mathcal{L}}$  satisfying properties (1)-(9).

We now describe some immediate consequences of some of these properties.

**Lemma 4.2.** *Suppose  $\underline{\mathcal{L}}$  satisfies (1)-(4). Then for all  $j < i$ ,  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_j) = 0$ .*

*Proof.* We proceed by induction on  $i - j$ . The base case holds by (2). Now suppose  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_j) = 0$  for some  $j < i$ , and apply  $\text{Hom}(\mathcal{L}_i, -)$  to the  $j - 1$ th Euler sequence. The induced sequence starts

$$0 \rightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_{j-1}) \rightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_j)^{l_j-1}.$$

Since the right term is zero so is  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{j-1})$ , as desired.  $\square$

**Lemma 4.3.** *Suppose  $\underline{\mathcal{L}}$  satisfies (1)-(4). Then the numbers  $l_i$  and  $r_i$  are nonzero.*

*Proof.* Suppose one of  $l_i$  or  $r_i$  was equal to zero. Then  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) = 0$ , so that the  $i$ th Euler sequence would imply that  $\mathcal{L}_i = 0$ . This contradicts the fact that  $\text{End}(\mathcal{L}_i) = D_i$ .  $\square$

**Lemma 4.4.** *Suppose  $\underline{\mathcal{L}}$  satisfies (1), (3) and (4).*

- (1) *If  $a \in D_i$ , then there exists a unique  $f_a \in \text{End}(\mathcal{L}_{i+1}^{l_i})$  such that the diagram*

$$\begin{array}{ccc} \mathcal{L}_i & \xrightarrow{h} & \mathcal{L}_{i+1}^{l_i} \\ a \downarrow & & \downarrow f_a \\ \mathcal{L}_i & \xrightarrow[h]{} & \mathcal{L}_{i+1}^{l_i} \end{array}$$

*whose horizontal arrows are from the  $i$ th Euler sequence, commutes.*

- (2) *The function  $a \mapsto f_a$  is a  $k$ -algebra homomorphism.*  
 (3) *There exists a  $k$ -algebra homomorphism*

$$\Phi_i : \text{End}(\mathcal{L}_i) \longrightarrow \text{End}(\mathcal{L}_{i+2})$$

*endowing  $\text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2})$  with an  $\text{End}(\mathcal{L}_i) - \text{End}(\mathcal{L}_{i+1})$ -bimodule structure.*

*Proof.* By assumption, the components of  $h = (h_1, \dots, h_{l_i})$  are a left basis for  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$ . Therefore, if  $a \in D_i$ , there exist  $a_{lj}$  in  $D_{i+1}$  such that

$$(4-2) \quad h_l a = \sum_j a_{lj} h_j.$$

We let  $f_a \in \text{End}(\mathcal{L}_{i+1}^{l_i})$  denote the morphism uniquely determined by the fact that if  $g_i : \mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i+1}^{l_i}$  denotes the  $i$ th inclusion, then  $f_a g_j = (a_{1j}, \dots, a_{l_i j})$ . Uniqueness of  $f_a$  follows from the fact that  $\{h_1, \dots, h_{l_i}\}$  is a left basis for  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$ . Part (1) follows.

The proof (2) is routine and omitted.

For (3), we define the  $k$ -algebra homomorphism  $\Phi_i : D_i \longrightarrow D_{i+2}$  as follows: given  $a \in D_i$ , part (1) implies that we get a unique  $f_a \in \text{End}(\mathcal{L}_{i+1}^{l_i})$  such that the diagram

$$\begin{array}{ccc} \mathcal{L}_i & \xrightarrow{h} & \mathcal{L}_{i+1}^{l_i} \\ a \downarrow & & \downarrow f_a \\ \mathcal{L}_i & \xrightarrow{h} & \mathcal{L}_{i+1}^{l_i} \end{array}$$

commutes. It follows that there is a unique  $a' \in D_{i+2}$  making the diagram

$$(4-3) \quad \begin{array}{ccc} \mathcal{L}_{i+1}^{l_i} & \longrightarrow & \mathcal{L}_{i+2} \\ \downarrow f_a & & \downarrow a' \\ \mathcal{L}_{i+1}^{l_i} & \longrightarrow & \mathcal{L}_{i+2} \end{array}$$

whose horizontal arrows are those in the  $i$ th Euler sequence, commute. We define  $\Phi_i(a) := a'$ . The proof that  $\Phi_i$  is a  $k$ -algebra homomorphism is routine and omitted.  $\square$

## 5. LINEAR SEQUENCES AND THEIR PROPERTIES

Throughout this section, we let  $\mathcal{C}$  denote a  $k$ -linear abelian category, and we let  $\underline{\mathcal{L}}$  be a sequence of objects in  $\mathcal{C}$ . We say that  $\underline{\mathcal{L}}$  is *linear* if it satisfies properties (1)-(5) from Section 4. The purpose of this section is to study properties of linear sequences and use them to begin the proof of Theorem 4.1.

### 5.1. Linear sequences.

**Proposition 5.1.** *Suppose  $\underline{\mathcal{L}}$  is linear. If, for each  $i$ , we endow  $\text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2})$  with the  $D_i - D_{i+1}$ -bimodule structure from Lemma 4.4(3), then there is an isomorphism of bimodules*

$$\Psi_i : {}^* \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \longrightarrow \text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2}).$$

*Proof.* Throughout the proof, in order to simplify the notation, we let  $l := l_i$  and we write  $\Psi$  instead of  $\Psi_i$ . We first construct  $\Psi$ . We let

$$0 \rightarrow \mathcal{L}_i \xrightarrow{h} \mathcal{L}_{i+1}^l \xrightarrow{h'} \mathcal{L}_{i+2} \rightarrow 0$$

denote the  $i$ th Euler sequence, and we write  $h = (h_1, \dots, h_l)$ . We let  $\{^*h_1, \dots, ^*h_l\}$  denote the right basis for  ${}^* \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$  dual to the left basis  $\{h_1, \dots, h_l\}$  of the  $D_{i+1} - D_i$ -bimodule  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$ . We define  $\Psi$  by letting it send  $^*h_m$  to  $h' \circ g_m$  where  $g_m$  is inclusion of the  $m$ th factor of  $\mathcal{L}_{i+1}$  in  $\mathcal{L}_{i+1}^l$ , and we extend right-linearly.

We need to show  $\Psi$  is one-to-one, onto, and compatible with left multiplication. We show first that  $\Psi$  is one-to-one. Suppose  $a_1, \dots, a_l \in D_{i+1}$  are such that

$$h'(\sum_j g_j a_j) = 0.$$

Then  $\sum_j g_j a_j : \mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i+1}^l$  factors through the kernel of  $h'$ . Since, by property (2) of  $\underline{\mathcal{L}}$ , we have  $\text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_i) = 0$ , it follows that  $a_1 = \dots = a_l = 0$ . Therefore,  $\Psi$  is one-to-one. To prove that  $\Psi$  is onto, it suffices to prove that the right dimension of  $\text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2})$  is equal to  $l$ . This follows from counting the right-dimensions of the terms in the long exact sequence resulting from applying the

functor  $\text{Hom}(\mathcal{L}_{i+1}, -)$  to the  $i$ th Euler sequence, and noting that  $\text{Ext}^1(\mathcal{L}_{i+1}, \mathcal{L}_i) = 0$  by property (2) of  $\underline{\mathcal{L}}$ .

We now show that  $\Psi$  is compatible with the left  $D_i$ -module structure. To this end, we suppose the right action on the left basis  $\{h_1, \dots, h_l\}$  is given by (4-2). Then, by [9, Lemma 3.4], the left action on the right basis  $\{^*h_1, \dots, ^*h_l\}$  of  $^*\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$  is given by

$$a^*h_r = \sum_s ^*h_s a_{sr}.$$

Therefore, by definition of  $\Psi$ , we have

$$\begin{aligned} \Psi(a \cdot (\sum_r ^*h_r b_r)) &= \Psi(\sum_{r,s} ^*h_s a_{sr} b_r) \\ &= h'(\sum_s g_s (\sum_r a_{sr} b_r)). \end{aligned}$$

On the other hand, if  $b_1, \dots, b_l \in D_{i+1}$ , then by definition of  $f_a$  from Lemma 4.4 we have

$$(5-1) \quad f_a(b_1, \dots, b_l) = (\sum_r a_{1r} b_r, \dots, \sum_r a_{lr} b_r).$$

Therefore,

$$\begin{aligned} a \cdot (\sum_r h'_r g_r b_r) &= h'(f_a(\sum_r g_r b_r)) \\ &= h'(\sum_s g_s (\sum_r a_{sr} b_r)). \end{aligned}$$

as desired.  $\square$

The following is an adaptation of [7, Proposition 2.2].

**Lemma 5.2.** *Suppose  $\underline{\mathcal{L}}$  is linear, and retain the notation from Proposition 5.1. Then, under the composition*

$$\begin{aligned} ^*\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \otimes \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) &\xrightarrow{\Psi_i \otimes 1} \text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2}) \otimes \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \\ &\longrightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+2}) \end{aligned}$$

whose second arrow is induced by composition, the element  $\sum_j ^*h_j \otimes h_j$  goes to zero.

*Proof.* As in the proof of Proposition 5.1, we let  $g_m : \mathcal{L}_{i+1} \longrightarrow \mathcal{L}_{i+1}^{\oplus l_i}$  denote the  $m$ th inclusion. By definition of  $\Psi_i$ , the element  $\sum_j ^*h_j \otimes h_j$  maps to  $h'(\sum_j g_j h_j)$ . But this is zero as it is the composition of maps in the  $i$ th Euler sequence.  $\square$

**Proposition 5.3.** *Suppose  $\underline{\mathcal{L}}$  is linear. Then the  $D_{i+1} - D_i$ -bimodule  $M := \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$  is admissible.*

*Proof.* The fact that the left duals of  $M$  are finite dimensional on either side follows immediately from Proposition 5.1. On the other hand, if  $N = \text{Hom}(\mathcal{L}_{i-1}, \mathcal{L}_i)$  then Proposition 5.1, implies that  $M \cong ^*N$ . Since  $N$  is finite-dimensional on either side by linearity of  $\underline{\mathcal{L}}$ , the canonical map  $N \rightarrow (^*N)^*$  is an isomorphism of bimodules, and so  $M^* \cong N$ . Since  $i$  is arbitrary, the result follows.  $\square$

In light of Lemma 5.2 and Proposition 5.3, the proof of the next result is similar to the proof of [18, Proposition 3.4]. We leave the details to the interested reader.



**Proposition 5.4.** *Suppose  $\underline{\mathcal{L}}$  is linear, let  $j \in \mathbb{Z}$ , and let  $M = \text{Hom}(\mathcal{L}_{-(j+1)}, \mathcal{L}_{-j})$ . Then, for all  $i \in \mathbb{Z}$ , there is a canonical isomorphism of bimodules*

$$\Psi_i : M^{i*} \xrightarrow{\cong} \text{Hom}(\mathcal{L}_{-j-(i+1)}, \mathcal{L}_{-j-i})$$

where the bimodule structure on  $\text{Hom}(\mathcal{L}_{-j-(i+1)}, \mathcal{L}_{-j-i})$  is given by the appropriate composition of maps  $\Phi_l$  defined in Lemma 4.4(3). Furthermore,

$$Q_i \subset \ker(\Psi_i \otimes \Psi_{i+1}).$$

**Corollary 5.5.** *Suppose  $\underline{\mathcal{L}}$  is linear, and let  $i \in \mathbb{Z}$ . Then  $H_{ii+1}$  is admissible and there is a  $k$ -linear  $\mathbb{Z}$ -algebra homomorphism*

$$(5-2) \quad \mathbb{S}^{nc}(H_{01}) \rightarrow H$$

which is an isomorphism in degrees zero and one.

*Proof.* By Proposition 5.3,  $H_{ii+1}$  is admissible. The rest of the result follows immediately from Proposition 5.4.  $\square$

**5.2. Proof of Theorem 4.1: Part 1.** In this section we retain the notation from Section 4. Our goal is to prove the first three parts of Theorem 4.1 as well as the forward direction of the fourth part of Theorem 4.1. We begin by noting that the first part of the theorem follows from Corollary 5.5. The second part of the theorem follows easily from the next result.

**Lemma 5.6.** *Suppose  $\underline{\mathcal{L}}$  is linear and satisfies property (6) from Section 4. Then, for each  $i \in \mathbb{Z}$  and  $j > i + 1$ , the map*

$$\text{Hom}(\mathcal{L}_{j-1}, \mathcal{L}_j) \otimes \cdots \otimes \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \rightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_j)$$

induced by composition is surjective.

*Proof.* Let  $j > i + 1$  and apply  $\text{Hom}(\mathcal{L}_i, -)$  to the  $j - 2$ th Euler sequence. Since  $\text{Ext}^1(\mathcal{L}_i, \mathcal{L}_{j-2}) = 0$  by property (6) of  $\underline{\mathcal{L}}$ , the induced map  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{j-1}^{l_{j-2}}) \rightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_j)$  is surjective. Therefore, the map

$$(5-3) \quad \text{Hom}(\mathcal{L}_{j-1}, \mathcal{L}_j) \otimes \text{Hom}(\mathcal{L}_i, \mathcal{L}_{j-1}) \rightarrow \text{Hom}(\mathcal{L}_i, \mathcal{L}_j)$$

induced by composition is surjective.

We now proceed to prove the lemma by induction on  $j$ . If  $j = i + 2$ , surjectivity of (5-3) implies the base case. If  $j > i + 2$ , surjectivity of (5-3) together with the induction hypothesis implies the result.  $\square$

**Proposition 5.7.** *If  $\underline{\mathcal{L}}$  is linear and satisfies (6) and (7), and  $j \in \mathbb{Z}$ , then*

- (1)  $H_{jj+1}$  is 2-periodic, and
- (2)  $H_{jj+1}$  is not of type  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$ .

*Proof.* To prove the first assertion, we note that  $H_{jj+1}$  is admissible by Corollary 5.5. Furthermore

$$\begin{aligned} \text{l dim } H_{jj+1}^{i*} &= \text{l dim } \text{Hom}(\mathcal{L}_{-j-(i+1)}, \mathcal{L}_{-j-i}) \\ &= l_{-j-(i+1)} \\ &= r_{-j-(i+2)} \\ &= \text{r dim } \text{Hom}(\mathcal{L}_{-j-(i+2)}, \mathcal{L}_{-j-(i+1)}) \\ &= \text{r dim } H_{jj+1}^{i+1*} \end{aligned}$$

where the first and last equality follow from Proposition 5.4, while the third equality follows from property (7) of  $\underline{\mathcal{L}}$ . Therefore,  $H_{jj+1}$  is 2-periodic.

We next show that if  $\underline{\mathcal{L}}$  satisfies properties (1)-(7), then  $H_{01}$  is not of type (1, 3). By 2-periodicity of  $H_{01}$ , it will follow that  $H_{jj+1}$  is not of type (1, 3). The rest of the proof is similar (but easier) and omitted.

Suppose  $H_{01}$  is of type (1, 3). We first assume  $l_{-1} = 1$  while  $r_{-1} = 3$ . Then by 2-periodicity of  $H_{01}$  and Proposition 5.4, if  $i$  has odd parity then  $l_i = 1$  and  $r_i = 3$ , while if  $i$  has even parity, then  $l_i = 3$  and  $r_i = 1$ .

Suppose, first, that  $i$  has odd parity. Then the  $-i$ th Euler sequence has the form

$$0 \rightarrow \mathcal{L}_{-i} \rightarrow \mathcal{L}_{-i+1} \rightarrow \mathcal{L}_{-i+2} \rightarrow 0.$$

Applying  $\text{Hom}(\mathcal{L}_{-i}, -)$ , we deduce, by property (6) of  $\underline{\mathcal{L}}$ , that

$$(5-4) \quad \text{rdim Hom}(\mathcal{L}_{-i}, \mathcal{L}_{-i+2}) = 2.$$

Next, applying  $\text{Hom}(\mathcal{L}_{-i}, -)$  to the  $-i + 1$ st Euler sequence

$$0 \rightarrow \mathcal{L}_{-i+1} \rightarrow \mathcal{L}_{-i+2}^3 \rightarrow \mathcal{L}_{-i+3} \rightarrow 0$$

and using (5-4) allows us to deduce, by property (6) of  $\underline{\mathcal{L}}$ , that

$$\text{rdim Hom}(\mathcal{L}_{-i}, \mathcal{L}_{-i+3}) = 3.$$

Similarly,  $\text{rdim Hom}(\mathcal{L}_{-i}, \mathcal{L}_{-i+4}) = 1$  and thus  $\text{rdim Hom}(\mathcal{L}_{-i}, \mathcal{L}_{-i+5}) = 0$ . But now applying  $\text{Hom}(\mathcal{L}_{-i}, -)$  to

$$0 \rightarrow \mathcal{L}_{-i+4} \rightarrow \mathcal{L}_{-i+5} \rightarrow \mathcal{L}_{-i+6} \rightarrow 0.$$

allows us to deduce that  $\text{Hom}(\mathcal{L}_{-i}, \mathcal{L}_{-i+4})$  is zero, a contradiction.

One obtains a similar contradiction in the case that  $i$  has even parity. □

In light of Proposition 5.7, the third part of Theorem 4.1 and the forward direction of the fourth part of Theorem 4.1 follow from the next

**Proposition 5.8.** *If  $\underline{\mathcal{L}}$  is linear and satisfies (6) and (7), then the homomorphism*

$$\mathbb{S}^{nc}(H_{01}) \rightarrow H$$

*from Corollary 5.5 is an isomorphism.*

*If  $\underline{\mathcal{L}}$  satisfies (1)-(9) then the isomorphism above induces an equivalence*

$$\mathbb{P}^{nc}(H_{01}) \rightarrow \mathcal{C}.$$

*Proof.* To prove the first result, it suffices, by the second part of Theorem 4.1, to show that for all  $i, j \in \mathbb{Z}$ , the right dimension of  $H_{ij}$  equals that of  $A_{ij} := \mathbb{S}^{nc}(H_{01})$ . By definition of the noncommutative symmetric algebra, and by properties (2) and (4) of  $\underline{\mathcal{L}}$ ,  $H_{ij}$  and  $A_{ij}$  have right-dimension zero if  $j < i$ . Furthermore, by Corollary 5.5, the result holds if  $j - i$  equals 0 or 1. Now we prove the result by induction on the difference of indices. By properties (4) and (6) of  $\underline{\mathcal{L}}$ , we have an exact sequence of right  $D_{-j}$ -modules

$$0 \rightarrow H_{ij} \rightarrow H_{i-1j}^{\oplus l_{-i}} \rightarrow H_{i-2j} \rightarrow 0.$$

On the other hand, by Corollary 3.5, we have an exact sequence of right- $A_{jj} \cong D_{-j}$ -modules

$$0 \rightarrow A_{ij} \rightarrow A_{i-1j}^{\oplus \text{rdim } A_{i-2i-1}} \rightarrow A_{i-2j} \rightarrow 0$$

since  $j \geq i + 2$ . Now,  $l_{-i} = r_{-i-1}$  by (7), while by Proposition 5.1,  $l_{-i} = r_{-i+1}$ , so that  $r_{-i-1}$  is the right-dimension of  $\text{Hom}(\mathcal{L}_{-i+1}, \mathcal{L}_{-i+2}) = H_{i-2i-1}$ . Finally, by Proposition 5.4,  $H_{i-2i-1} \cong A_{i-2i-1}$ . The first result now follows by induction.

Now we prove the second result. Suppose  $\underline{\mathcal{L}}$  satisfies (1)-(9). Since (8) and (9) hold, the indicated equivalence follows from the first part and Theorem 2.2.  $\square$

**5.3. The finite-type case.** We say  $\underline{\mathcal{L}}$  is of *finite-type* if it satisfies (1) and each  $D_i$  is finite-dimensional over  $k$ .

Theorem 1.2 follows from the next

**Lemma 5.9.** *Suppose  $\underline{\mathcal{L}}$  is of finite-type and  $\dim_k D_i = \dim_k D_{i+2}$  for all  $i$ . Suppose further that  $\underline{\mathcal{L}}$  satisfies properties (2), (4), and (8). Then  $\underline{\mathcal{L}}$  satisfies properties (3), (5), and (7).*

*Proof.* By property (8),  $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$  is finite-dimensional on the right. Hence, it is finite-dimensional over  $k$  on the left and therefore finite-dimensional over  $D_{i+1}$  on the left so that (3) holds.

Property (5) holds since, by Lemma 4.4(3),  $\Phi_i$  is a  $k$ -algebra injection and, by hypothesis,  $\dim_k D_i = \dim_k D_{i+2}$ .

Finally, we check (7). Let  $m_i$  denotes the dimension of  $D_i$  over  $k$ . Then

$$\begin{aligned} r_i m_i &= l_i m_{i+1} \\ &= r_{i+1} m_{i+1} \\ &= l_{i+1} m_{i+2}, \end{aligned}$$

where the second equality follows from Proposition 5.1. The result follows.  $\square$

The next result follows immediately.

**Corollary 5.10.** *Suppose  $\underline{\mathcal{L}}$  is such that  $\text{End}(\mathcal{L}_i) = k$  for all  $i$ , and satisfies (2), (4) and (8). Then  $\underline{\mathcal{L}}$  satisfies (3), (5) and (7).*

## 6. INTERNAL HOM FUNCTORS AND THEIR DERIVED FUNCTORS

To prove Theorem 4.1, it remains to complete the proof of the fourth part of the theorem. To this end, we show that noncommutative  $\mathbb{P}^1$ -bundles are homologically well-behaved. More specifically, in Section 7, we prove that if  $D_0$  and  $D_1$  are division rings over  $k$ ,  $M$  is a  $k$ -central  $D_0 - D_1$ -bimodule such that  $M$  is 2-periodic and not of type (1, 1), (1, 2) or (1, 3), then  $\mathbb{S}^{nc}(M)$  is Gorenstein (Corollary 7.3). This will allow us to prove homological results about  $\mathbb{P}^{nc}(M)$ , Corollary 7.5, Theorem 7.6, and Lemma 7.7 that we will need to complete the proof of Theorem 4.1.

In this section we will describe many of the homological preliminaries we will need in Section 7: in Section 6.1 we recall the definition and basic properties of the internal Hom functor introduced in [19]. Since we work here over an affine base, the constructions in [19] simplify considerably. For this reason, and for the convenience of the reader, we will develop the material from first principals. Next, in Section 6.2, we study the right-derived functors of the internal Hom functor.

Throughout this section, if  $R$  is a ring,  $\text{Mod} R$  will denote the category of right  $R$ -modules. Furthermore, unless otherwise stated,  $A$  will denote a  $\mathbb{Z}$ -algebra such that, for all  $i$ ,  $A_{ii}$  is a division ring with  $k$  in its center. To define the internal Hom functor, we will need the following

**Definition 6.1.** Let  $\text{Bimod} A - A$  denote the category defined as follows:

- an object of  $\text{Bimod}A - A$  is a triple

$$(C = \{C_{ij}\}_{i,j \in \mathbb{Z}}, \{\mu_{ijk}\}_{i,j,k \in \mathbb{Z}}, \{\psi_{ijk}\}_{i,j,k \in \mathbb{Z}})$$

where  $C_{ij}$  is an  $A_{ii} - A_{jj}$ -bimodule and  $\mu_{ijk} : C_{ij} \otimes A_{jk} \rightarrow C_{ik}$  and  $\psi_{ijk} : A_{ij} \otimes C_{jk} \rightarrow C_{ik}$  are morphisms of  $A_{ii} - A_{kk}$ -bimodules making  $C$  both a graded right and left  $A$ -module such that the actions are compatible in the usual sense.

- A morphism  $\phi : C \rightarrow D$  between objects in  $\text{Bimod}A - A$  is a collection  $\{\phi_{ij}\}_{i,j \in \mathbb{Z}}$ , where  $\phi_{ij} : C_{ij} \rightarrow D_{ij}$  are morphisms of  $A_{ii} - A_{jj}$ -bimodules which respect the  $A - A$ -bimodule structure on  $C$  and  $D$ .

We omit the routine verification that  $\text{Bimod}A - A$  is abelian.

**6.1. Internal Hom.** We begin by defining the internal Hom functor. For  $C$  an object in  $\text{Bimod}A - A$  and  $M$  an object in  $\text{Gr}A$ ,

- we let

$$\text{Hom}_{\text{Gr}A}(e_i C, M)$$

denote the right  $A_{ii}$ -module with underlying set  $\text{Hom}_{\text{Gr}A}(e_i C, M)$  and with  $A_{ii}$ -action induced by the left action of  $A_{ii}$  on  $e_i C$ , and

- we let

$$\underline{\text{Hom}}_{\text{Gr}A}(C, M)$$

denote the object in  $\text{Gr}A$  with  $i$ th component  $\text{Hom}_{\text{Gr}A}(e_i C, M)$  and with multiplication induced by left-multiplication of  $A$  on  $C$ .

**Lemma 6.2.** *The assignment*

$$\underline{\text{Hom}}_{\text{Gr}A}(-, -) : (\text{Bimod}A - A)^{op} \times \text{Gr}A \rightarrow \text{Gr}A$$

*is a bifunctor in the sense of [13, Chapter 20.8]. Furthermore,*

- (1) *for each  $M$  in  $\text{Gr}A$ , there is a natural isomorphism  $\underline{\text{Hom}}_{\text{Gr}A}(A, M) \cong M$ , and*
- (2)  *$\underline{\text{Hom}}_{\text{Gr}A}(-, -)$  is left exact in each variable, and for each injective  $I$  in  $\text{Gr}A$ ,  $\underline{\text{Hom}}_{\text{Gr}A}(-, I)$  is exact.*

*Proof.* The fact that  $\underline{\text{Hom}}_{\text{Gr}A}(-, -)$  is a bifunctor follows easily from the fact that  $\text{Hom}_{\text{Gr}A}(-, -)$  is a bifunctor. The proof of the first item is routine while the second follows from properties of  $\text{Hom}_{\text{Gr}A}(-, -)$ .  $\square$

If  $C$  is an object of  $\text{Bimod}A - A$ , then we define two associated functors

$$- \otimes_{A_{ii}} e_i C : \text{Mod}A_{ii} \rightarrow \text{Gr}A,$$

and

$$\text{Hom}_{\text{Gr}A}(e_i C, -) : \text{Gr}A \rightarrow \text{Mod}A_{ii}.$$

As one can check, the pair  $(- \otimes_{A_{ii}} e_i C, \text{Hom}_{\text{Gr}A}(e_i C, -))$  has a canonical adjoint structure. This is exploited in the proof of the following

**Theorem 6.3.** *Suppose  $F$  is an  $A_{jj} - A_{ii}$ -bimodule of finite dimension on either side, and let  $F \otimes_{A_{ii}} e_i C$  denote the object of  $\text{Bimod}A - A$  such that*

$$(F \otimes_{A_{ii}} e_i C)_{lm} = \begin{cases} F \otimes_{A_{ii}} C_{im} & \text{if } l = j \\ 0 & \text{otherwise,} \end{cases}$$

*endowed with the obvious bimodule structure. Let  $M$  be an object of  $\text{Gr}A$ . Then*

- (1) *there is a natural isomorphism of  $\text{Mod } A_{jj}$ -valued functors*

$$\mathcal{H}om_{\text{Gr } A}(F \otimes e_i C, -) \cong \mathcal{H}om_{\text{Gr } A}(e_i C, -) \otimes_{A_{ii}} F^*,$$

and

- (2) *the functor  $\mathcal{H}om_{\text{Gr } A}(F \otimes e_i A, -) : \text{Gr } A \rightarrow \text{Mod } A_{jj}$  is exact.*

*Proof.* By adjointness, there is a canonical isomorphism

$$\text{Hom}_{\text{Gr } A}(F \otimes e_i C, M) \rightarrow \text{Hom}_{\text{Mod } A_{ii}}(F, \mathcal{H}om_{\text{Gr } A}(e_i C, M))$$

where  $F$  is considered as an  $A_{ii}$ -module. Furthermore, as one can check, this isomorphism is compatible with right  $A_{jj}$ -module structure. Finally, by the Eilenberg-Watts theorem, there is a natural isomorphism

$$\text{Hom}_{\text{Mod } A_{ii}}(F, \mathcal{H}om_{\text{Gr } A}(e_i C, M)) \rightarrow \mathcal{H}om_{\text{Gr } A}(e_i C, M) \otimes_{A_{ii}} F^*$$

of right  $A_{jj}$ -modules, completing the proof of the first assertion.

The proof of the second assertion follows from the first, in light of the fact that the functor  $-\otimes_{A_{ii}} F^* : \text{Mod } A_{ii} \rightarrow \text{Mod } A_{jj}$  is exact (see, for example, [22, Section 2.1]), and that  $\mathcal{H}om_{\text{Gr } A}(e_i A, -) \cong (-)_i$ .  $\square$

**6.2. Internal Ext.** Let  $C \in \text{Bimod } A - A$ . In this section we study the right derived functors of  $\underline{\mathcal{H}om}_{\text{Gr } A}(C, -)$  and  $\mathcal{H}om_{\text{Gr } A}(C, -)$ . The fact that  $\underline{\mathcal{H}om}_{\text{Gr } A}(C, -)$  and  $\mathcal{H}om_{\text{Gr } A}(C, -)$  have right derived functors follows from Lemma 6.2(2). We denote them by  $\underline{\mathcal{E}xt}_{\text{Gr } A}^i(C, -)$  and  $\mathcal{E}xt_{\text{Gr } A}^i(C, -)$ . We note that since taking the  $j$ th degree part of an object of  $\text{Gr } A$  is an exact functor from  $\text{Gr } A$  to  $\text{Mod } A_{jj}$ , we have

$$(\underline{\mathcal{E}xt}_{\text{Gr } A}^i(C, M))_j \cong \mathcal{E}xt_{\text{Gr } A}^i(e_j C, M).$$

In order to state the next lemma, we need to introduce some terminology. We say an object  $C$  of  $\text{Bimod } A - A$  is *left-bounded by degree  $l$*  if for each  $i$ ,  $C_{ij} \neq 0$  implies  $j \geq i + l$ . For  $n$  a nonnegative integer, we let  $A_{\geq n}$  denote the subobject of  $A$  in  $\text{Bimod } A - A$  given by  $\oplus_{j-i \geq n} A_{ij}$ .

**Lemma 6.4.** *For  $C \in \text{Bimod } A - A$  and  $M \in \text{Gr } A$ ,*

- (1) *the sequence  $\underline{\mathcal{E}xt}_{\text{Gr } A}^i(-, M)$  forms a  $\delta$ -functor,*
- (2) *if  $M$  is right-bounded by degree  $r$  and  $C$  is left-bounded by degree  $l$ , then  $\underline{\mathcal{E}xt}_{\text{Gr } A}^i(C, M)$  is right-bounded by degree  $r - l$ , and*
- (3) *if  $M$  is right-bounded then for all  $j \geq 1$ , the right bound of  $\underline{\mathcal{E}xt}_{\text{Gr } A}^j(A/A_{\geq n}, M)$  tends to  $-\infty$  as  $n \rightarrow \infty$ .*

*Proof.* The first result follows directly from [13, Proposition 8.4, p. 810] in light of Lemma 6.2(2).

To prove the second result, we adapt the proof of [4, Proposition 3.1(2)] to our context. To this end, we claim that  $\mathcal{E}xt_{\text{Gr } A}^i(e_j C, M) = 0$  if and only if  $\text{Ext}_{\text{Gr } A}^i(e_j C, M) = 0$ . This follows from the fact that if  $i_* : \text{Mod } A_{jj} \rightarrow \text{Mod } k$  is the restriction of scalars functor, then  $i_* \mathcal{H}om_{\text{Gr } A}(e_j C, -) \cong \text{Hom}_{\text{Gr } A}(e_j C, -)$ . The claim implies that the argument in the proof of [4, Proposition 3.1(2)] may be used to prove the second result.

The third result is the  $\mathbb{Z}$ -algebra version of [4, Proposition 3.1(5)], and we recount the proof for the convenience of the reader. By part (1), the functor  $\underline{\mathcal{H}om}_{\text{Gr } A}(-, M)$  applied to the short exact sequence in  $\text{Bimod } A - A$

$$(6-1) \quad 0 \rightarrow A_{\geq n} \rightarrow A \rightarrow A/A_{\geq n} \rightarrow 0$$

induces a long-exact sequence. Since, by Lemma 6.2(1),  $\underline{\mathcal{H}om}_{\text{Gr}A}(A, -)$  is exact, it thus suffices to show that, for  $i \geq 0$ ,  $\underline{\mathcal{E}xt}_{\text{Gr}A}^i(A_{\geq n}, M)$  is right-bounded by  $r - n$ , where  $r$  is the right-bound of  $M$ . This is exactly part (2) so the result follows.  $\square$

**Lemma 6.5.** *Suppose  $F$  is an  $A_{jj} - A_{ii}$ -bimodule of finite dimension on either side, and let  $F \otimes_{A_{ii}} e_i C$  denote the object of  $\text{Bimod}A - A$  defined in Theorem 6.3. If  $M$  be an object of  $\text{Gr}A$ , then there is a natural isomorphism of  $\text{Mod}A_{jj}$ -valued functors*

$$\mathcal{E}xt_{\text{Gr}A}^j(F \otimes e_i C, -) \cong \mathcal{E}xt_{\text{Gr}A}^j(e_i C, -) \otimes F^*.$$

*Proof.* The  $j = 0$  case follows from Theorem 6.3(1).

By Lemma 6.4(1), and the fact that the composition of a  $\delta$ -functor with an exact functor is a  $\delta$ -functor, the sequence  $\mathcal{E}xt_{\text{Gr}A}^j(-, M)$  is a  $\delta$ -functor. Furthermore, for  $M$  injective and  $j > 0$ ,  $\mathcal{E}xt_{\text{Gr}A}^j(-, M) = 0$  by Lemma 6.2(2). The result now follows from [10, Theorem 1.3A, p. 206].  $\square$

Suppose  $\lambda, \rho \in \mathbb{Z}$  with  $\lambda \leq \rho$  and let  $M \in \text{Gr}A$ . We write  $M \subset [\lambda, \rho]$  if  $M_i$  nonzero implies that  $\lambda \leq i \leq \rho$ . Similarly, suppose  $l, r \in \mathbb{Z}$  with  $l \leq r$  and let  $C \in \text{Bimod}A$ . We write  $C \subset [l, r]$  if  $C_{ij}$  nonzero implies  $l \leq j - i \leq r$ . We say  $C$  is concentrated in degree  $m$  if  $C \subset [m, m]$ . We let  $A_0$  denote the quotient  $A/A_{\geq 1}$  in  $\text{Bimod}A - A$ .

**Lemma 6.6.** *Let  $i$  be a nonnegative integer, let  $C \in \text{Bimod}A$  be such that  $C \subset [l, r]$ , and let  $M \in \text{Gr}A$  be such that*

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^j(A_0, M) \subset [\lambda, \rho]$$

*for all  $j \geq i$ . Then, for  $j \geq i$ ,*

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^j(C, M) \subset [\lambda - r, \rho - l].$$

*Proof.* First, assume  $C$  is concentrated in degree  $m$ . By Lemma 6.5,

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^i(C, M)_n \cong \underline{\mathcal{E}xt}_{\text{Gr}A}^i(A_0, M)_{n+m} \otimes C_{n, n+m}^*.$$

Thus, since  $\underline{\mathcal{E}xt}_{\text{Gr}A}^j(A_0, M) \subset [\lambda, \rho]$  for  $j \geq i$ , we have  $\underline{\mathcal{E}xt}_{\text{Gr}A}^j(C, M) \subset [\lambda - m, \rho - m]$  for  $j \geq i$ .

Now suppose  $C \subset [l, r]$  and define a subobject,  $C'$  of  $C$  by letting  $e_n C' = C_{n, n+r}$ . Then we have an exact sequence

$$(6-2) \quad 0 \rightarrow C' \rightarrow C \rightarrow C/C' \rightarrow 0.$$

Furthermore, by construction,  $C/C' \subset [l, r - 1]$ . We now prove the result by induction on  $r - l$ , the case  $r = l$  being proven above.

Suppose the result holds when  $r - l < m$  and let  $C \in \text{Bimod}A$  be such that  $C \subset [l, r]$ , with  $r - l = m$ . The long exact sequence for  $\underline{\mathcal{H}om}_{\text{Gr}A}(-, M)$  applied to (6-2) contains

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^j(C/C', M) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr}A}^j(C, M) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr}A}^j(C', M).$$

If  $j \geq i$ , then the left module is contained in  $[\lambda - r + 1, \rho - l]$ , whereas the right module is in  $[\lambda - r, \rho - r]$ , so that the assertion follows.  $\square$

For the remainder of the section, we suppose that  $D_0$  and  $D_1$  are division rings over  $k$ , and  $A = \mathbb{S}^{nc}(M)$  where  $M$  is a 2-periodic  $D_0$ - $D_1$ -bimodule over  $k$  not of type  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$ . Then  $A$  satisfies the conditions in Section 2.1, so that  $\text{Proj}A$ , as well as the functors  $\pi, \tau$ , and  $\omega$ , are defined.

**Lemma 6.7.** *There is an isomorphism of functors  $\tau(-) \cong \lim_{n \rightarrow \infty} \underline{\mathcal{H}om}_{\text{Gr}A}(A/A_{\geq n}, -)$ .*

*Proof.* Let  $M \in \text{Gr}A$ . By Lemma 6.2(1) the canonical map  $A \rightarrow A/A_{\geq n}$  induces an inclusion

$$\lim_{n \rightarrow \infty} \underline{\mathcal{H}om}_{\text{Gr}A}(A/A_{\geq n}, M) \rightarrow M$$

natural in  $M$ . It is a routine verification to prove that the image of this map is exactly  $\tau(M)$  and the result follows easily from this.  $\square$

It follows from Corollary 3.5 and Theorem 6.3(2) that for all  $n \geq 0$  and all  $j \in \mathbb{Z}$ ,  $e_j A/e_j A_{\geq n}$  has a finite resolution by objects  $C \in \text{Bimod}A - A$  such that  $\mathcal{H}om_{\text{Gr}A}(e_j C, -)$  is exact. It thus follows from [13, Proposition 8.2, p. 809] that  $\mathcal{E}xt_{\text{Gr}A}^i(e_j A/e_j A_{\geq n}, -)$  can be computed as the  $n$ th cohomology of  $\mathcal{H}om_{\text{Gr}A}(-, -)$  applied to this resolution.

**Theorem 6.8.** *Let  $N$  be a graded  $A$ -module. for  $j \geq 1$ , we have*

$$R^j \omega(\pi(N)) \cong \lim_{n \rightarrow \infty} \underline{\mathcal{E}xt}_{\text{Gr}A}^j(A_{\geq n}, N).$$

*Furthermore, for  $i \geq 1$ , the right-derived functors of  $\tau$  and  $\omega$  satisfy*

$$R^{i+1} \tau(N) \cong R^i \omega(\pi(N)),$$

*and, for each  $N \in \text{Gr}A$ , there is an exact sequence*

$$0 \rightarrow \tau N \rightarrow N \rightarrow \omega \pi N \rightarrow R^1 \tau N \rightarrow 0,$$

*whose central arrow is natural.*

*Proof.* We adapt the proof of [5, Lemma 4.1.5] to our context. We first claim, as in [5, Lemma 4.1.3], that for  $T \in \text{Gr}A$  a torsion module, we have  $R^i \tau(T) = 0$  for  $i > 0$ . To prove the claim, it suffices, by Lemma 6.7, to prove that

$$\lim_{n \rightarrow \infty} \mathcal{E}xt_{\text{Gr}A}^i(e_j A/e_j A_{\geq n}, T) = 0$$

for  $i > 0$  and all  $j$ . By the comments preceding the theorem, the functor

$$\lim_{n \rightarrow \infty} \mathcal{E}xt_{\text{Gr}A}^i(e_j A/e_j A_{\geq n}, -)$$

commutes with direct limits, so that we may assume, without loss of generality, that  $T$  is right-bounded by degree  $r$ . The claim now follows from Lemma 6.4(3).

We next claim that  $\omega \pi N \cong \lim_{n \rightarrow \infty} \underline{\mathcal{H}om}_{\text{Gr}A}(A_{\geq n}, N)$ . To prove this, we note that by the first claim, and by the fact that  $A$  satisfies the properties from Section 2.1, we can copy the proof of [5, Lemma 4.1.4].

Now we prove the theorem. The first statement is just the second claim. The second statement comes the long exact sequence constructed by applying

$$\lim_{n \rightarrow \infty} \underline{\mathcal{H}om}_{\text{Gr}A}(-, N)$$

to the short exact sequence (6-1).  $\square$

## 7. RELATIVE LOCAL COHOMOLOGY OF NONCOMMUTATIVE SYMMETRIC ALGEBRAS

In this section we show that noncommutative symmetric algebras are Gorenstein (Theorem 7.1), and apply this result to the computation of the right derived functors of the torsion functor  $\tau$  (Corollary 7.3). This allows us to compute certain cohomology groups over  $\mathbb{P}^{nc}(M)$ , and these computations are used to complete the proof of Theorem 4.1 in Section 7.2.

We assume throughout this section that  $D_0$  and  $D_1$  are division rings over  $k$ ,  $M$  is a 2-periodic  $D_0$ - $D_1$   $k$ -central bimodule not of type  $(1, 1)$ ,  $(1, 2)$  or  $(1, 3)$ , and  $A = \mathbb{S}^{nc}(M)$ .

**7.1.  $\mathbb{S}^{nc}(M)$  is Gorenstein.** By the remarks preceding Theorem 6.8, the derived functors of  $\mathcal{H}om_{\text{Gr}A}(A/A_{\geq 1}, -)$  and  $\underline{\mathcal{H}om}_{\text{Gr}A}(A/A_{\geq 1}, -)$  may be computed using the Euler sequence from Corollary 3.5. This fact will be utilized in the proof of the next

**Theorem 7.1.** *Let  $i \geq 0$  and let  $l$  and  $j$  be integers. Then*

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^i(A/A_{\geq 1}, e_l A) = 0 \text{ for } i \neq 2$$

and

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq 1}, e_l A)_j \cong \begin{cases} A_{l-2, l-2} & \text{if } j = l-2, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By the comment preceding the theorem, we may compute

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^i(A/A_{\geq 1}, e_l A)_m$$

by taking the cohomology of the sequence  
(7-1)

$$\mathcal{H}om_{\text{Gr}A}(e_m A, e_l A) \xrightarrow{d_0} \mathcal{H}om_{\text{Gr}A}(A_{mm+1} \otimes e_{m+1} A, e_l A) \xrightarrow{d_1} \mathcal{H}om_{\text{Gr}A}(Q_m \otimes e_{m+2} A, e_l A)$$

coming from the application of  $\mathcal{H}om_{\text{Gr}A}(-, e_l A)$  to the truncation of the exact sequence

$$Q_m \otimes e_{m+2} A \rightarrow A_{mm+1} \otimes e_{m+1} A \rightarrow e_m A$$

from Corollary 3.5. In particular, we have

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^0(A/A_{\geq 1}, e_l A)_m = \ker d_0,$$

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^1(A/A_{\geq 1}, e_l A)_m = \frac{\ker d_1}{\text{im } d_0},$$

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq 1}, e_l A)_m = \frac{\mathcal{H}om_{\text{Gr}A}(Q_m \otimes e_{m+2} A, e_l A)}{\text{im } d_1}$$

and  $\underline{\mathcal{E}xt}_{\text{Gr}A}^i(A/A_{\geq 1}, e_l A)_m = 0$  for  $i > 2$ .

If  $m < l-2$  it follows from Theorem 6.3 that all terms in (7-1) are zero so that the indicated groups are zero. Similarly, if  $m = l-2$ , the first two terms of (7-1) vanish. Therefore,  $\underline{\mathcal{E}xt}_{\text{Gr}A}^0(A/A_{\geq 1}, e_l A)_{l-2} = \underline{\mathcal{E}xt}_{\text{Gr}A}^1(A/A_{\geq 1}, e_l A)_{l-2} = 0$  and

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq 1}, e_l A)_{l-2} \cong A_{ll} = A_{l-2, l-2}$$

by Theorem 6.3. Therefore, to establish the theorem, we must prove that the sequence (7-1) is exact for  $m > l-2$ .



We first show  $\ker d_0 = 0$ . If  $m = l - 1$ , this holds since the left-most term of (7-1) is zero. Next, suppose  $f : e_m A \rightarrow e_l A$ , and let

$$\mu : A_{mm+1} \otimes e_{m+1} A \rightarrow e_m A$$

denote multiplication. If  $m = l$ , then either  $f = 0$  or  $f$  is an isomorphism. In the latter case, since multiplication  $\mu$  is nonzero,  $d_0(f) \neq 0$ . Therefore, the result holds in this case. Thus, we suppose  $m > l$ . If  $d_0(f) = 0$ , then since  $\mu$  is surjective in degree  $\geq m + 1$ , it suffices to show that if  $f$  is left-multiplication by  $x \in A_{lm}$ , then  $xy = 0$  for all  $y \in A_{mm+1}$  implies that  $x = 0$ . This follows from Lemma 3.6.

Next, we prove that if  $m > l - 2$ , then  $\ker d_1 = \text{im } d_0$ . It suffices to prove that  $\ker d_1 \subset \text{im } d_0$ . Suppose  $g : A_{mm+1} \otimes e_{m+1} A \rightarrow e_l A$  is in  $\ker d_1$ . Then  $g$  has a factorization

$$A_{mm+1} \otimes e_{m+1} A \xrightarrow{\mu} (e_m A)_{\geq m+1} \xrightarrow{f} e_l A.$$

Thus, to complete the proof of the result in this case, we must show that  $f$  extends to a right  $A$ -module map  $\tilde{f} : e_m A \rightarrow e_l A$ . Suppose  $x_1, \dots, x_n$  is a right basis for  $A_{mm+1}$  with associated right duals  $x_1^*, \dots, x_n^* \in A_{m+1m+2}$ . Then

$$\begin{aligned} \sum_p f(x_p) x_p^* &= f\left(\sum_p x_p x_p^*\right) \\ &= 0. \end{aligned}$$

It thus follows from Theorem 3.4 that  $\sum_p f(x_p) \otimes x_p^* \in A_{lm+1} \otimes A_{m+1m+2}$  is an element of the image of  $A_{lm} \otimes Q_m$  under the left map of (3-5). Therefore, there exists a  $y \in A_{lm}$  such that, for all  $p$ ,  $f(x_p) = y x_p$ . We let  $\tilde{f}$  be defined by  $\tilde{f}(1) = y$ , so that  $\tilde{f}$  extends  $f$  as desired.

Finally, we prove that if  $m > l - 2$ , then  $\text{im } d_1 = \text{Hom}_{\text{Gr} A}(Q_m \otimes e_{m+2} A, e_l A)$ . To this end, given a right  $A$ -module morphism  $f : Q_m \otimes e_{m+2} A \rightarrow e_l A$ , we must show that it factors as  $Q_m \otimes e_{m+2} A \rightarrow A_{mm+1} \otimes e_{m+1} A \xrightarrow{g} e_l A$ . With the notation as in the previous paragraph, we note that  $f(\sum_p x_p \otimes x_p^* \otimes 1)$  is in  $A_{lm+2}$  so has the form  $\sum_p y_p x_p^*$  for some  $y_1, \dots, y_n \in A_{lm+1}$ . We define  $g$  by letting  $g(x_p \otimes 1) = y_p$ .  $\square$

**Corollary 7.2.** *If  $n \geq 1$  and  $M$  is in  $\text{Gr} A$ , then  $\underline{\mathcal{E}xt}_{\text{Gr} A}^i(A/A_{\geq n}, M) = 0$  for  $i > 2$  and  $\underline{\mathcal{E}xt}_{\text{Gr} A}^i(A/A_{\geq n}, e_l A) = 0$  for  $i \neq 2$ .*

*Proof.* We prove the result by induction on  $n$ . When  $n = 1$ , the first result follows from the remark immediately preceding Theorem 7.1, while the second result follows from Theorem 7.1.

For the general case, we note that, by Lemma 6.4(1), the exact sequence

$$0 \rightarrow A_{\geq n}/A_{\geq n+1} \rightarrow A/A_{\geq n+1} \rightarrow A/A_{\geq n} \rightarrow 0$$

in  $\text{Bimod} A$  induces a long exact sequence, of which

$$\underline{\mathcal{E}xt}_{\text{Gr} A}^i(A/A_{\geq n}, M) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr} A}^i(A/A_{\geq n+1}, M) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr} A}^i(A_{\geq n}/A_{\geq n+1}, M)$$

is a part. If  $i > 2$  the left term is zero by induction while the right term is zero by Lemma 6.5 and induction. Therefore, the center is zero in this case. If  $i = 0$  or  $i = 1$  and  $M = e_l A$ , the same reasoning ensures that the center is zero.  $\square$

**Corollary 7.3.** *Suppose  $M$  is an object of  $\text{Gr} A$ .*

(1) *For  $i > 2$ ,*

$$R^i \tau(M) = 0.$$

(2) For  $i \neq 2$

$$R^i \tau(e_l A) = 0.$$

(3)

$$(R^2 \tau(e_l A))_{l-2-i} = \begin{cases} A_{l-2-i, l-2}^* & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* By Lemma 6.7 and the fact that  $\text{Gr}A$  has exact direct limits, the first two results follow directly from Corollary 7.2.

To prove (3), we prove two preliminary results. We first claim that

$$\underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n+1}, e_l A) \subset [l-2-n, l-2].$$

To this end, we note that, by Corollary 7.2, the sequence

$$0 \rightarrow A_{\geq n}/A_{\geq n+1} \rightarrow A/A_{\geq n+1} \rightarrow A/A_{\geq n} \rightarrow 0$$

induces an exact sequence

$$(7-2) \quad 0 \rightarrow \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n}, e_l A) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n+1}, e_l A) \rightarrow \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A_{\geq n}/A_{\geq n+1}, e_l A) \rightarrow 0.$$

If  $n = 1$  in (7-2), then the claim follows from Lemma 6.6, Theorem 7.1 and Lemma 6.5. The general case follows from the induction hypotheses and (7-2).

We next claim that  $\underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n+1}, e_l A)_{l-2-n} \cong A_{l-2-n, l-2}^*$ . To prove this, we note that when  $n = 0$ , the claim follows from Theorem 7.1. For  $n > 0$ ,

$$\begin{aligned} \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n+1}, e_l A)_{l-2-n} &\cong \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A_{\geq n}/A_{\geq n+1}, e_l A)_{l-2-n} \\ &\cong A_{l-2-n, l-2}^* \end{aligned}$$

where the first isomorphism follows from the first claim and (7-2), while the second isomorphism follows from the  $n = 0$  case and Lemma 6.5.

Finally, we prove (3). We have

$$\begin{aligned} (R^2 \tau(e_l A))_{l-2-i} &\cong \lim_{n \rightarrow \infty} \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq n}, e_l A)_{l-2-i} \\ &\cong \underline{\mathcal{E}xt}_{\text{Gr}A}^2(A/A_{\geq i+1}, e_l A)_{l-2-i} \\ &\cong A_{l-2-i, l-2}^*, \end{aligned}$$

where the first isomorphism is from Lemma 6.7, the second isomorphism follows from the first claim, and the third isomorphism follows from the second claim.  $\square$

For the rest of Section 7, in addition to our previous assumptions on  $A$ ,

- we assume  $A$  is coherent,
- for each  $i \in \mathbb{Z}$ , we let  $\mathcal{A}_i$  denote  $\pi(e_i A) \in \text{cohproj}A$ , and
- if  $\underline{\mathcal{L}} : \text{cohproj}A \rightarrow \text{Proj}A$  is the functor defined in Lemma 2.1, we abuse notation by letting  $\mathcal{A}_i$  denote  $\underline{\mathcal{L}}(\pi(e_i A))$ .

**Lemma 7.4.** *The functor  $\underline{\mathcal{L}}$  from Lemma 2.1 induces an isomorphism of  $\text{End } \mathcal{A}_j - \text{End } \mathcal{A}_i$ -bimodules*

$$(7-3) \quad \text{Ext}_{\text{cohproj}A}^1(\mathcal{A}_j, \mathcal{A}_i) \rightarrow \text{Ext}_{\text{Proj}A}^1(\mathcal{A}_j, \mathcal{A}_i).$$

*Proof.* Suppose we have an extension

$$(7-4) \quad 0 \rightarrow \mathcal{A}_i \rightarrow \mathcal{P} \rightarrow \mathcal{A}_j \rightarrow 0$$

in  $\text{Proj} A$ . Without loss of generality, we may suppose that  $\mathcal{P} = \pi P$  where  $P$  is a torsion-free object of  $\text{Gr} A$ . Since, by Theorem 6.8 and Corollary 7.3(2), there is a natural isomorphism  $e_l A \rightarrow \omega \mathcal{A}_l$ , applying  $\omega$  to (7-4) yields an exact sequence

$$0 \rightarrow e_i A \rightarrow \omega \mathcal{P} \rightarrow e_j A \rightarrow R^1 \omega(\mathcal{A}_i)$$

in  $\text{Gr} A$ . By Theorem 6.8 and Corollary 7.3(3), the image of the right map is bounded, hence coherent. It follows that  $\omega \mathcal{P}$  is coherent, so that we may assume, without loss of generality, that  $\mathcal{P}$  is the image, under  $\pi : \text{Gr} A \rightarrow \text{Proj} A$ , of a coherent module. The fact that (7-4) is in the image of (7-3) now follows from Lemma 2.1(3) and Corollary 7.3(2). Thus, (7-3) is surjective. The injectivity of (7-3) follows from Lemma 2.1(4) and Corollary 7.3(2).  $\square$

For the next two results, we use the fact that there are isomorphisms

$$\begin{aligned} \text{Hom}_{\text{Proj} A}(\mathcal{A}_j, \pi(-)) &\cong \text{Hom}_{\text{Gr} A}(e_j A, \omega \pi(-)) \\ &\cong (\omega \pi(-))_j. \end{aligned}$$

**Corollary 7.5.** *There are isomorphisms of  $\text{End } \mathcal{A}_i - \text{End } \mathcal{A}_j$ -bimodules*

$$\text{Ext}_{\text{cohproj} A}^q(\mathcal{A}_j, \mathcal{A}_i) \cong \begin{cases} A_{ij} & \text{if } q = 0 \\ A_{ji-2}^* & \text{if } q = 1 \end{cases}.$$

*Proof.* If  $q = 0, 1$ , there are isomorphisms of  $\text{End } \mathcal{A}_j - \text{End } \mathcal{A}_i$ -bimodules

$$\text{Ext}_{\text{cohproj} A}^q(\mathcal{A}_j, \mathcal{A}_i) \cong \text{Ext}_{\text{Proj} A}^q(\mathcal{A}_j, \mathcal{A}_i)$$

by Lemma 2.1(2) and (3) and Lemma 7.4. In addition, by the remark preceding the corollary, for  $q = 0, 1$  there are isomorphisms

$$\text{Ext}_{\text{cohproj} A}^q(\mathcal{A}_j, \mathcal{A}_i) \cong (R^q \omega(\mathcal{A}_i))_j.$$

Furthermore, by Corollary 7.3(2),  $\tau(e_l A) = R^1 \tau(e_l A) = 0$ . Thus, by Theorem 6.8,

$$\text{Hom}_{\text{cohproj} A}(\mathcal{A}_j, \mathcal{A}_i) \cong A_{ij},$$

and, by Theorem 6.8 and Corollary 7.3(3),

$$\begin{aligned} \text{Ext}_{\text{cohproj} A}^1(\mathcal{A}_j, \mathcal{A}_i) &\cong (R^2 \tau(e_i A))_j \\ &\cong A_{ji-2}^*. \end{aligned}$$

$\square$

We now prove a generalization of Serre vanishing [20, Theorem 3.5(2)].

**Theorem 7.6.** *If  $M$  is a coherent object of  $\text{Gr} A$ ,*

$$\text{Ext}_{\text{Proj} A}^1(\mathcal{A}_i, \pi(M)) = 0$$

*whenever  $i \gg 0$ .*

*Proof.* By the remark preceding Corollary 7.5, there is an isomorphism

$$\text{Ext}_{\text{Proj} A}^1(\mathcal{A}_i, \pi M) \rightarrow R^1 \omega(\pi(M))_i.$$

By Theorem 6.8, the right-hand side is isomorphic to  $(R^2 \tau M)_i$ . Since  $R^3 \tau = 0$  by Corollary 7.3(1), it suffices, by considering a presentation of  $M$ , to show that, for any  $j$ ,  $(R^2 \tau e_j A)_i = 0$  for  $i \gg 0$ . This follows immediately from Corollary 7.3.  $\square$

**Lemma 7.7.** *If  $M$  is coherent, then  $(R^i \tau M)_j$  and  $(R^i \omega(\pi(M)))_j$  are finite-dimensional over  $A_{jj}$  for all  $i \geq 0$ .*

*Proof.* We first show that  $(R^i \tau M)_j$  is finite-dimensional over  $A_{jj}$  for all  $i \geq 0$ . By Corollary 7.3(2) and (3), the result holds when  $M \cong e_{i_1} A \oplus \cdots \oplus e_{i_n} A$ . Next, we note that  $M$  coherent implies that there is a short exact sequence in  $\mathbf{Gr} A$

$$0 \rightarrow R \rightarrow \bigoplus_{l \in I} e_l A \rightarrow M \rightarrow 0$$

where  $I$  is finite and  $R$  is coherent. Therefore, by Corollary 7.3(1), the first result follows from descending induction on  $i$  as in the proof of [20, Lemma 3.2].

The second part of the lemma follows from the first part and Theorem 6.8.  $\square$

## 7.2. Proof of Theorem 4.1: Part 2.

**Theorem 7.8.** *The sequence  $(\mathcal{A}_{-i})_{i \in \mathbb{Z}}$  in  $\mathbf{cohproj} A$  satisfies properties (1)-(9) from Section 4.*

*Proof.* Properties (1), (2), (3), and (6) follow immediately from Corollary 7.5, while property (7) follows from the fact that  $A_{01}$  is 2-periodic. Property (4) follows from (3-8) since the right-dimension of  $A_{-i-2, -i-1}$  equals the left-dimension of  $A_{-i-1, -i}$ . Property (5) follows from Proposition 3.7.

Next, we check property (8). By the remark preceding Corollary 7.5, there are  $A_{jj}$ -vector space isomorphisms

$$\mathrm{Hom}_{\mathbf{Proj} A}(\mathcal{A}_j, \pi M) \cong \mathrm{Hom}_{\mathbf{Gr} A}(e_j A, \omega \pi M) \cong (\omega \pi M)_j.$$

Therefore, property (8) follows from Lemma 2.1(2) and Lemma 7.7.

It remains to check property (9). We first show the indicated sequence is projective in the sense of Section 2.3. To this end, we first claim that if  $f : \mathcal{N} \rightarrow \mathcal{M}$  is an epimorphism in  $\mathbf{cohproj} A$  and  $\mathcal{N}$  is a finite direct sum of modules of the form  $\mathcal{A}_l$ , then there exists an  $n$  such that  $\mathrm{Hom}_{\mathbf{cohproj} A}(\mathcal{A}_i, f)$  is surjective for all  $i > n$ . By [8, Corollaire 1, p. 368] and Corollary 7.3(2), we may assume that  $\ker f = \pi K$  where  $\iota(K)$  is torsion-free. By applying  $\mathrm{Hom}_{\mathbf{cohproj} A}(\mathcal{A}_i, -)$  to this sequence, the claim is reduced to showing that there exists an  $n$  such that for all  $i > n$ ,  $\mathrm{Ext}_{\mathbf{cohproj} A}^1(\mathcal{A}_i, \pi(K)) = 0$ . This last fact follows from Theorem 7.6 and Lemma 2.1(4).

For the general case, we use the fact that there is an epimorphism from a finite direct sum of modules of the form  $\mathcal{A}_i$  to  $\mathcal{N}$ , which induces an epimorphism,  $g$ , to  $\mathcal{M}$ . The claim then implies that there is some  $n$  such that for all  $i > n$ ,  $\mathrm{Hom}_{\mathbf{cohproj} A}(\mathcal{A}_i, g)$  is surjective, which then implies that  $\mathrm{Hom}_{\mathbf{cohproj} A}(\mathcal{A}_i, f)$  is surjective, as desired.

Finally, the fact that our sequence is ample follows from the definition of coherence and property (4).  $\square$

## 8. AN APPLICATION

In this section we confirm the  $\mathbb{P}_n^1$  has a sequence  $\underline{\mathcal{L}}$  satisfying the properties (1)-(9) of Section 4. Corollary 1.3 follows as an immediate consequence. Throughout the section, we let

$$A = k\langle x_1, \dots, x_n \rangle / (b),$$

where  $x_i$  has degree one for all  $i$ , and  $b = \sum_{i=1}^n x_i \sigma(x_{n-i})$  for some graded automorphism  $\sigma$  of the free algebra. Since  $A$  is coherent by [24, Theorem 4.3],  $\mathbf{cohproj} A$  is abelian, and  $\mathbf{cohproj} A$  is  $k$ -linear by [3, Proposition B8.1]. For  $j \in \mathbb{Z}$ , we let  $[j]$  denote the shift functor on  $\mathbf{Gr} A$ , so that  $M[j]_i := M_{i+j}$ . We let  $\pi : \mathbf{coh} A \rightarrow \mathbf{cohproj} A$

denote the quotient functor, and we define

$$\pi(A[i]) := \mathcal{L}_i.$$

By the proof of [24, Proposition 5.1(2)],  $\text{End } \mathcal{L}_i = k$ . Therefore, by Corollary 5.10, it suffices to confirm properties (2), (4), (6), (8) and (9) of  $\underline{\mathcal{L}}$ . By the proof of [24, Proposition 5.1(2)], the first equality of (2) holds. By [24, Proposition 1.5(b)],  $\mathbb{P}_n^1$  satisfies Serre duality (see [29, Section 1.3] for the exact form of the duality). The second part of (2), as well as property (6), follows from duality. Property (4) follows from [29, Section 1.9]. Property (8) follows from [24, Proposition 5.1(3)]. Finally, property (9) is observed at the end of the statement of [24, Proposition 5.1].

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